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Quantization of Modified Maxwell's Electrodynamics

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The work presented in this Thesis is, to the best of my knowledge and belief original, except as acknowledged in the text, and has not been submitted either in whole or in part, for a degree at this or any other university.

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Abstract

The standard model of particle physics is incomplete, with unexplained phenomena such as dark matter, quantum gravity and matter-antimatter asymmetry remaining unsolved despite extensive research. The most common searches for undiscovered physics occur through the addition of hypothetical particles to the standard model. A less common route in this endeavour is the introduction of fundamental interactions between known particles. The photon does not interact with itself in the standard model. However, a new nonlinear model called Modified Maxwell's electrodynamics, or *ModMax* for short, has been discovered in theoretical literature that predicts a photon capable of interacting with itself without breaking the symmetries of Maxwell's theory.

ModMax has been studied extensively at the semi-classical level with applications in strongly coupled condensed matter systems, however remains untouched in a quantum context. As such, it was the central aim of this project to perform the quantization of this theory. Using the background field method and dimensional regularization, I obtained novel corrections beyond what the classical theory predicts. By calculating the effective action, I showed that these corrections vanish in a constant background field, and are not of the form of the classical theory for a varying background field.

Motivated by the form of the corrections obtained for ModMax, I applied the method I developed to quantize ModMax to its two dimensional analogue theory. This was the secondary aim of the project, and I obtained the effective action by evaluating all one loop Feynman diagrams, as well as the separate infinite series of two vertex diagrams. Lastly, I considered an alternative approach to quantization using auxiliary fields that captured the nonlinearity, and demonstrated that it is not possible to quantize ModMax in this fashion without breaking Lorentz symmetry. As few theories of nonlinear electrodynamics have been explored on the quantum domain, this quantization of ModMax represents a forward step in this endeavour. While ModMax's theoretical effects remain on a scale unreachable by current experimental techniques, I nonetheless characterized ModMax's predictions and it's analogue's behaviour on the quantum domain.

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List of Symbols

The following list is neither exhaustive nor exclusive, but may be helpful.

d	number of spacetime dimensions (usually $1+3$)
μ, ν, ρ, τ ...	4-vector indices running $(0,1,2,3)$
i, j, k	3-vector indices running $(1,2,3)$
$g^{\mu\nu}$	metric tensor (always Minkowski or Euclidean)
$\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$	real numbers, natural numbers, and integers respectively
$\mathcal{O}\left(\gamma^n\right)$	up to order γ^n, higher powers are discarded

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Review: Introduction

1.1 Preface

While common extensions to the standard model in the search for new physics often add new theoretical particles, a less common route in this endeavour is the addition of novel interactions between known particles. An emerging category of such extensions is models of *nonlinear electrodynamics*, that is, adding self-interactions of photons with themselves. Such interactions break the principle of superposition that arises from the linearity of Maxwell's equations and thus are significant only at extreme scales.

These models have been extensively studied at the classical level to solve problems in cosmology and supergravity as well as in strongly coupled condensed matter systems where photon self-interactions would contribute significantly [1]. However, they remain largely unstudied in the quantum domain due to the difficulty the nonlinearity introduces to quantization procedures.

There are a number of ways to extend electrodynamics while preserving different properties of interest within classical electromagnetism. The prototypical example of such an extension is the Born-Infeld theory, proposed in 1934 to solve the infinite self-energy of an electron [2–5]. Born and Infeld achieved this by introducing a maximum possible electric field strength in their theory. However this modification introduces a characteristic energy scale (the maximum field strength) which breaks the scale invariance present in Maxwell's electrodynamics. Note that Born-Infeld theory still preserves a symmetry of Maxwell's equations called electromagnetic duality.

It was long thought that there were no possible extensions to Maxwell's electrodynamics that would preserve both of the present symmetries: scale invariance and electromagnetic duality. In recent literature [1, 6, 7] however, a novel modification to Maxwell's theory of electromagnetism (electrodynamics) was discovered that achieves this: *Modified Maxwell's electrodynamics* or *ModMax* for short. It was further proved that ModMax is the only theory that achieves this, namely, it is the unique nonlinear extension that preserves all the symmetries of Maxwell's original theory. The beauty inherent in the unique preservation of these symmetries aside, ModMax is also of particular interest as such symmetries can lead to novel observable implications when the theory is quantized.

Additionally, as we expect such symmetries to be respected in classical limits, it is of great interest whether such symmetries are fundamental or broken at the quantum level. While the domain of effect of such extensions is beyond current experimental techniques, the presence of nonlinear effects represents a conceptual shift in how we describe electromagnetism worthy of our study.

1.2 Introduction

In this thesis, we perform the quantization of ModMax and calculate such first quantum corrections that arise within this theory. We also generalize our argument to other higher derivative theories in $1 + 1$ spacetime dimensions.

The process of quantization, the transfer of a classical theory to the quantum domain, begins with formulating a Lagrangian, an object which completely specifies the theory and the equations of motion it predicts. Often this is identical to the classical Lagrangian, now with the fields being considered quantum fields that do not commute (i.e. order matters in an expression).

If the Lagrangian describes non-interacting particles, then it is often able to be solved exactly for the equations of motion. However, among interacting theories, very few admit an exact solution and thus we must employ the use of perturbation theory [8, 9]. To achieve this, we consider the interaction to have a small effect relative to the free evolution of the particles and expand in an increasing number of interactions. This is well suited to nonlinear theories of electrodynamics where the additional interaction term is separable, as the strength of the interaction is necessarily extremely small due to lack of classical observation.

However, due to the nonlinearity of ModMax and the non-analytic nature of the square root it contains, perturbation theory alone cannot yield a quantum version of this theory. In addition, we must expand the nonlinearity itself about some background. In most nonlinear field theories, the weak field limit reduces to Maxwell's equations, which would allow one to truncate higher powers of fields in such an expansion immediately. However, ModMax only reduces to Maxwell's theory in the *non-interacting limit*, namely when the self-interaction terms vanish, which is distinct. Instead, we make use of the *background field method*, where we expand about a fixed classical background field. This fixed classical background provides a valid point to expand about, and will reduce correctly in the limit to Maxwell's equations.

1.3 Classical Electromagnetism

Beginning with Maxwell's theory, Maxwell's Lagrangian is expressible as

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{1.1}$$

where the field strength $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.2}$$

for four vector potential A_μ, with $\mu = 0, 1, 2, 3$. We can also write the electric and magnetic fields explicitly with derivatives of this potential such that for $i = 1, 2, 3$ we have

$$E_i = \partial_0 A_i - \partial_i A_0 \qquad\qquad B_i = -\varepsilon_{ijk} \partial^j A^k, \tag{1.3}$$

where we use Einstein notation in which summation over repeated indices is implied.

Definition 1: The **Hodge dual** of the field strength tensor $F_{\mu\nu}$ is defined as

$$\widetilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\tau} F_{\rho\tau}, \tag{1.4}$$

where $\varepsilon^{\mu\nu\rho\tau} = -\varepsilon^{\mu\nu\tau\rho}$ is the Levi-Civita tensor that is antisymmetric under all index exchanges.

Applying the Euler-Lagrange equation,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu\right)} = \frac{\partial \mathcal{L}}{\partial A_\nu}, \tag{1.5}$$

leads to the equations of motion

$$\partial_\mu F^{\mu\nu} = 0 \qquad \partial_\mu \widetilde{F}^{\mu\nu} = 0, \tag{1.6}$$

which can be written in the more familiar form

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} &= \boldsymbol{\nabla} \times \mathbf{B} & \frac{\partial \mathbf{B}}{\partial t} &= -\boldsymbol{\nabla} \times \mathbf{E}, \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 & \boldsymbol{\nabla} \cdot \mathbf{E} &= 0, \end{aligned} \tag{1.7}$$

which are the familiar (sourceless) Maxwell's equations.

1.4 Symmetries of Maxwell's Equations

Maxwell's equations has two symmetries of note that are preserved uniquely by ModMax: electromagnetic duality and conformal invariance.

One can notice that under an $SO(2)$ transformation (i.e. a 2D rotation) parametrised by an angle $\alpha \in [0, 2\pi)$,

$$\begin{pmatrix} F'^{\mu\nu} \\ \widetilde{F}'^{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} F^{\mu\nu} \\ \widetilde{F}^{\mu\nu} \end{pmatrix}, \tag{1.8}$$

that Maxwell's equations of motion are invariant. This is a symmetry called *electromagnetic duality* (EM-duality) that Maxwell's theory possesses and is the reason we consider electric and magnetic fields as part of a larger electromagnetic theory. This duality is most apparent when performing Lorentz transformations, where electric and magnetic fields 'rotate' into each other in a similar fashion. Notice that this symmetry holds only *on-shell*, that is, it occurs when the equations of motion are applied. ModMax preserves this symmetry at this level as well [1, 7].

Writing the field strength tensor and its dual explicitly we see the element wise exchange (up to sign) of electric and magnetic fields with

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad \widetilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{1.9}$$

Additionally, Maxwell's theory has no dependence on a length or energy scale and thus has a global symmetry of scale invariance. With the addition of special conformal transformations, this becomes *conformal invariance*. This is a global symmetry of the Lagrangian, not just the equations of motion which is important as it allows us to apply Noether's theorem and derive conserved quantities. This group of symmetries includes all transformations that preserve angles and thus includes the Poincaré group (which is Lorentz transformations and 3D spatial rotations), dilations (zooming in/out) and special conformal transformations. The latter two transform the coordinates according to

$$x^\mu \to \lambda x^\mu \tag{1.10}$$

$$x^\mu \to \frac{x^\mu - \lambda^\mu x^2}{1 - 2\lambda_\mu x^\mu + \lambda^2 x^2}, \tag{1.11}$$

where λ^μ parametrizes the transformation.

More generally, in Minkowski space where the metric is given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.12}$$

we can write an infinitesimal distance as

$$\mathrm{d}^2 s = \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu\,. \tag{1.13}$$

Under a conformal transformation parametrized by $\Omega(x)$, this distance transforms as

$$\mathrm{d}^2 s' = e^{\Omega(x)}\, \mathrm{d}^2 s\,, \tag{1.14}$$

which preserves the relative angles of vectors (as shown in Fig. 1.1).

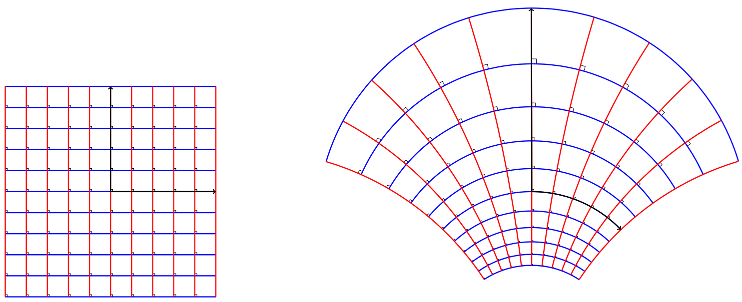


Figure 1.1: A special conformal transformation of a grid. Notice that the right angle intersections of all grid lines is preserved after the transformation.

Note. The special conformal transformation can also be interpreted as a coordinate inversion composed with a translation and then a second coordinate inversion.

1.5 General Theories of Nonlinear Electrodynamics

When investigating extensions to Maxwell's Lagrangian, there are strong restrictions on the form of the candidates and their nonlinearity. The theory must be Lorentz invariant to agree with experimental observations, and thus must be built out of Lorentz invariant operators. The only two independent Lorentz invariant combinations of operators that can be made with the field strength are the Maxwell Lagrangian

$$S \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \tag{1.15}$$

which is a scalar, and

$$P \equiv -\frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu} = \mathbf{E}\cdot\mathbf{B}, \tag{1.16}$$

which is a pseudo-scalar (i.e. negates under parity transformations). All higher order combinations of $F^{\mu\nu}$, such as $F^{\mu\nu}F_{\nu}{}^{\rho}F_{\rho\mu}$, are expressible in terms of these two invariants, and terms including more derivatives, such as $\partial^{\mu}S\partial_{\mu}S$ lead to systems associated with unphysical 'ghost fields' [1, 10].

Therefore, the most general form of a nonlinear electrodynamics Lagrangian we focus on is some function of these quantities, $\mathcal{L}\left(S,P\right)$. Note that this form contains no restrictions on the symmetries of the theory and in general can break both the symmetries of Maxwell's equations: conformal symmetry and electromagnetic duality. However, in an effort to narrow the search scope, we can identify that if we want our nonlinear extension to maintain conformal invariance, it must transform under a rescaling (by a constant a) of $S \to a^{-4}S$ and $P \to a^{-4}P$ as

$$\mathcal{L}\left(a^{-4}S, a^{-4}P\right) = a^{-4}\mathcal{L}\left(S,P\right). \tag{1.17}$$

This factor of a^{-4} is cancelled by the transformation of $\mathrm{d}^4x \to a^4\,\mathrm{d}^4x$ in the action integral to leave the theory invariant under this transformation.

Likewise, if we want our nonlinear extension to maintain electromagnetic duality, it's equations of motion should be invariant under the generalization of the $SO\left(2\right)$ rotation we saw in Eq. (1.8) for Maxwell's equations,

$$\begin{pmatrix} -2\frac{\partial\mathcal{L}(F')}{\partial F'_{\mu\nu}} \\ \widetilde{F}'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} \\ \widetilde{F}_{\mu\nu} \end{pmatrix}. \tag{1.18}$$

1.6 Classical ModMax

In recent work [6], it was shown that there is a unique family of Lagrangians which satisfy these two constraints and thus preserve these two symmetries of Maxwell's equations. This family of Lagrangians is ModMax, and is a family of theories as it satisfies these conditions for any real value of a dimensionless constant $\gamma \in \mathbb{R}$ that parametrizes the family.

As expected, we can write the ModMax Lagrangian in terms of the two Lorentz invariants S and P with

$$\mathcal{L} = S\cosh\gamma + \sinh\gamma\sqrt{S^2 + P^2} \tag{1.19}$$

or equivalently, using the definitions of S and P,

$$= -\frac{\cosh\gamma}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\sinh\gamma}{4}\sqrt{\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \left(F_{\mu\nu}\widetilde{F}^{\mu\nu}\right)^2} \tag{1.20}$$

$$= -\frac{\cosh\gamma}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) + \frac{\sinh\gamma}{2}\sqrt{\left(\mathbf{E}^2 - \mathbf{B}^2\right)^2 + 4\left(\mathbf{E}\cdot\mathbf{B}\right)^2}, \tag{1.21}$$

where γ is interpreted as the dimensionless coupling constant that determines the strength of the nonlinear self-interaction in the second term. Notice that when $\gamma = 0$ we recover Maxwell's Lagrangian as the nonlinear term disappears as $\sinh(0) = 0$.

While this family of Lagrangians possesses the symmetries of Maxwell's equations for all values of $\gamma \in \mathbb{R}$, for $\gamma < 0$ the theory predicts faster than light propagation of photons which violate causality. Therefore, we take $\gamma > 0$ for which causality is preserved [1, 6].

Applying the Euler-Lagrange equations, we find the equations of motion of the theory to be

$$\cosh\gamma\partial_\mu F^{\mu\nu} + \sinh\gamma\partial_\mu\left(\frac{SF^{\mu\nu} + P\widetilde{F}^{\mu\nu}}{\sqrt{S^2 + P^2}}\right) = 0, \tag{1.22}$$

$$\Rightarrow \partial_\mu F^{\mu\nu} = \tanh\gamma\partial_\mu\left(\frac{SF^{\mu\nu} + P\widetilde{F}^{\mu\nu}}{\sqrt{S^2 + P^2}}\right). \tag{1.23}$$

Notice that while these equations are nonlinear, if the field satisfies $P = aS$ for constant $a \in \mathbb{R}$, then they linearise.

Further, as ModMax preserves the conformal symmetry of Maxwell's equations, it is also invariant under conformal transformations. However, more commonly, one makes use of the fact that conformal invariance implies that the stress energy tensor of the theory is traceless.

Theorem 1: Given a conformally invariant Lagrangian $\mathcal{L}$, the **stress energy tensor** [11] defined by

$$T_{\mu\nu} = -2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}, \tag{1.24}$$

is *traceless* such that

$$T^{\mu}{}_{\mu} = 0. \tag{1.25}$$

Applying this to ModMax, we find that the stress energy tensor can be written as

$$T_{\mu\nu} = -2\left(\frac{\partial \mathcal{L}}{\partial S}\frac{\partial S}{\partial g^{\mu\nu}} + \frac{\partial \mathcal{L}}{\partial P}\frac{\partial P}{\partial g^{\mu\nu}}\right) + g_{\mu\nu}\mathcal{L}, \tag{1.26}$$

where

$$\frac{\partial S}{\partial g^{\mu\nu}} = -\frac{1}{2}F_{\mu}{}^{\rho}F_{\nu\rho} \qquad \frac{\partial P}{\partial g^{\mu\nu}} = -\frac{1}{4}\left(F_{\mu}{}^{\rho}\widetilde{F}_{\nu\rho} + F_{\nu}{}^{\rho}\widetilde{F}_{\mu\rho}\right). \tag{1.27}$$

In fact we have

$$T^{\mu\nu} = \left(F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}\left(F_{\rho\tau}F^{\rho\tau}\right)\right)\frac{\partial \mathcal{L}}{\partial S} \tag{1.28}$$

where

$$\frac{\partial \mathcal{L}}{\partial S} = \cosh\gamma - \sinh\gamma\frac{F_{\mu\nu}F^{\mu\nu}}{\sqrt{\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \left(F_{\mu\nu}\widetilde{F}^{\mu\nu}\right)^2}}. \tag{1.29}$$

From Eq. (1.28), we see immediately that $T^{\mu}{}_{\mu} = 0$, and thus ModMax is conformal.

1.7 Experimental Observability of ModMax

Despite the lack of study at the quantum level, classical analysis of ModMax indicates that it predicts a refractive index of the vacuum $n \neq 1$[1]. This is not unexpected as the nonlinearities that arise within the standard model also predict a vacuum refractive index differing from $n = 1$.

The most precise experimental test of the nonlinearity of the vacuum was recently conducted by the PVLAS experiment (*Polarizzazione del Vuoto con LASer*, "polarization of the vacuum with laser"). Using a cavity with mirrors, their experiment attempts to observe any interaction of light with itself or with the vacuum (i.e. spontaneous pair production). Due to the extremely small scale of any present nonlinearity, they were able to obtain an upper bound of $\gamma \leq 3 \times 10^{-22}$ (dimensionless) with lower bound experiments currently underway [12]. This suggests that if ModMax is an accurate description of our universe's electrodynamics, it's nonlinear contribution is very small.

Specifically, the PVLAS experiment proceeds by measuring the birefringence of the vacuum through how far its refractive index differs from $n = 1$ with ModMax predicting

$$\Delta n_{\mathrm{ModMax}} = e^{\gamma} - 1 \approx \gamma. \tag{1.30}$$

While PVLAS observed a difference in refractive index of $\Delta n_{\mathrm{Obs}} \leq 3 \times 10^{-22}$ (dimensionless) [12], QED predicts $\Delta n_{\mathrm{QED}} \sim 4 \times 10^{-24}$ [1] and thus there remains two orders of magnitude of parameter space for ModMax to have observable contributions.

2

Review: Quantization of Quantum Electrodynamics

Prior to considering ModMax in a quantum context, we review the quantization of *quantum electrodynamics* (QED) and the techniques applicable to such a theory: with and without interactions with matter. We also review the effective action which will prove instrumental in the quantization of ModMax.

2.1 Quantum Electrodynamics (QED)

Maxwell's Lagrangian translates smoothly to it's quantum counterpart, the Quantum Electrodynamics (QED) Lagrangian. Considering only free photons (i.e. no matter/electrons), we can describe electromagnetic waves with an identical Lagrangian to Maxwell's with

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.1}$$

where this leads to an action S given by

$$\begin{aligned} S[A] &= \int \mathrm{d}^4 x \, \mathcal{L} && (2.2) \\ &= \frac{1}{2} \int \mathrm{d}^4 x \, A_\mu(x) \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) A_\nu(x) \, . && (2.3) \end{aligned}$$

Note that with the Fourier transform of this field, given by

$$A_\mu(k) = \int \mathrm{d}^4x\, e^{ik_\nu x^\nu} A_\mu(x)\,, \tag{2.4}$$

the action integral can be written as

$$S[A] = \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4k_1\,\mathrm{d}^4k_2\, e^{k_1^\alpha x_\alpha} A_\mu(k_1)\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right) e^{ik_2^\beta x_\beta} A_\nu(k_2) \tag{2.5}$$

where evaluating the derivatives on the exponential yields

$$= \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4k_1\,\mathrm{d}^4k_2\, e^{k_1^\alpha x_\alpha} A_\mu(k_1)\left(-k_2^2 g^{\mu\nu} + k_2^\mu k_2^\nu\right) e^{ik_2^\beta x_\beta} A_\nu(k_2) \tag{2.6}$$

and grouping the exponentials,

$$= \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4k_1\,\mathrm{d}^4k_2\, A_\mu(k_1)\left(-k_2^2 g^{\mu\nu} + k_2^\mu k_2^\nu\right) e^{i(k_1+k_2)\cdot x} A_\nu(k_2) \tag{2.7}$$

reveals that with $\delta^4(k_1+k_2) = \int \mathrm{d}^4x\, e^{i(k_1+k_2)\cdot x}$, integrating over x yields

$$= \frac{1}{2}\int \mathrm{d}^4k_1\,\mathrm{d}^4k_2\, A_\mu(k_1)\left(-k_2^2 g^{\mu\nu} + k_2^\mu k_2^\nu\right) A_\nu(k_2)\,\delta^4(k_1+k_2) \tag{2.8}$$

where integrating over k_2 absorbs the δ function and enforces $k \equiv k_1 = -k_2$ leaving

$$S[A] = \frac{1}{2}\int \mathrm{d}^4k\, A_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right) A_\nu(-k)\,. \tag{2.9}$$

This form of the action will prove useful.

2.2 Functional Integrals

In quantum field theory, there is an analogue of the partition function Z from statistical mechanics called the generating functional $Z[J]$ which depends on an arbitrary external source $J(x)$. $J(x)$ is the analogue of an external magnetic field B. The generating functional is a convenient albeit abstract method to determine correlation functions.

Definition 2: For the electromagnetic field, the **generating functional** [8, 9] is given by

$$Z\left[J\right] = \int \mathcal{D}A \exp\left(iS\left[A\right] + i\int \mathrm{d}^4x\, J_\mu\left(x\right) A^\mu\left(x\right)\right), \tag{2.10}$$

where $\int \mathcal{D}A$ is a *functional integral*, that is, it integrates over all possible functions or field configurations $A^\mu\left(x\right)$ can take. One can think of this as the continuous analogue of a sum over all possible states that a system can take as in the partition function.

Namely, taking derivatives of the generating functional yields **correlation functions** such that

$$\left\langle A_{\mu_1}\left(x_1\right)\cdots A_{\mu_n}\left(x_n\right)\right\rangle = \left(-i\right)^n \left.\frac{\delta^n Z\left[J\right]}{\delta J_{\mu_1}\left(x_1\right)\cdots\delta J_{\mu_n}\left(x_n\right)}\right|_{J=0}, \tag{2.11}$$

where the nth order correlation function $\left\langle A_{\mu_1}\left(x_1\right)\cdots A_{\mu_n}\left(x_n\right)\right\rangle$ can be used to obtain probability amplitudes for a given interaction or decay process. One can also represent correlation functions in terms of Feynman diagrams as we will see.

However, one notices that the action $S\left[A\right]$ in Eq. (2.9) vanishes for all potentials $A_\mu\left(k\right) = k_\mu\alpha\left(k\right)$ where $\alpha\left(k\right)$ is any scalar function as

$$S\left[\alpha k\right] = -\frac{1}{2}\int \mathrm{d}^4k\,\alpha^2 k_\mu\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)k_\nu \tag{2.12}$$

$$= -\frac{1}{2}\int \mathrm{d}^4k\,\alpha^2\left(-k^4 + k^4\right) \tag{2.13}$$

$$= 0. \tag{2.14}$$

This is problematic for the theory as the partition function evaluated at $J = 0$ (i.e. no external source) leads to

$$Z\left[0\right] = \int \mathcal{D}A \exp\left(iS\left[A\right]\right) \tag{2.15}$$

$$= \int \mathcal{D}A e^0 \tag{2.16}$$

$$= \int \mathcal{D}A 1, \tag{2.17}$$

which diverges as there are uncountably infinite different possible field configurations $A^\mu\left(x\right)$ can take. This divergence in fact arises due to a lack of uniqueness in this theory's description of a given physical field configuration.

Claim. Namely, one can shift $A_\mu(x)$ by

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)\,, \tag{2.18}$$

for an arbitrary function $\alpha(x)$, without changing the physical implications of the theory. This shift is called a **gauge transformation**.

Note. We suppress the x dependence of α for brevity.

Proof. The field strength tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ transforms under this shift to

$$F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu\alpha) - \partial_\nu(A_\mu + \partial_\mu\alpha) \tag{2.19}$$

where as partial derivatives commute, yields

$$= \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.20}$$

$$= F_{\mu\nu}. \tag{2.21}$$

As the equations of motion, or equivalently the electric and magnetic fields can be written in terms of $F_{\mu\nu}$, the field configuration $A_\mu + \partial_\mu\alpha$ has identical physical implications and dynamics to A_μ. □

More generally we notice that the action itself is invariant under this transformation (as it can be written purely in terms of $F_{\mu\nu}$ as $S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$) such that

$$S[A_\mu] = S[A_\mu + \partial_\mu\alpha]\,. \tag{2.22}$$

This is referred to as a **gauge degree of freedom**, and is remedied by *fixing the gauge* which means we only count physically distinct states.

2.3 Faddeev-Popov Gauge Fixing

The cleanest way to achieve this gauge fixing in a path integral approach is through a method pioneered by Faddeev and Popov [13].

Definition 3: A Lagrangian $\mathcal{L}[A]$ has **local gauge symmetry** if it is invariant under a gauge transformation

$$A_\mu(x) \to A_\mu + \partial_\mu \alpha(x), \tag{2.23}$$

where $\alpha(x)$ is again an arbitrary function. Local here refers to the spacetime dependence of $\alpha(x)$. If it were constant $\alpha(x) = C \in \mathbb{R}$, it would be a **global symmetry**.

To fix the gauge, we define a gauge fixing function $G(A)$ that is zero for only one of every physical/gauge inequivalent state. It can be chosen to take the form

$$G(A) \equiv \partial_\mu A^\mu - \omega(x), \tag{2.24}$$

such that $G(A) = 0$ for only $\partial_\mu A^\mu = \omega(x)$. If the functional integral contained $\delta(G(A))$, then this would select only unique physical states.

Note. Under composition by a function $g : \mathbb{R}^n \to \mathbb{R}^n$, the delta function satisfies

$$\int \mathrm{d}^n x \, \delta^n(g(x)) \, f(g(x)) \det(\partial_j g_i) = \int \mathrm{d}^n x \, \delta^n(x) \, f(x). \tag{2.25}$$

If we integrate over a (infinite dimensional) space of functions $\mathcal{D}\alpha$ rather than $\mathbb{R}^n$, the analogous identity is

$$\int \mathcal{D}\alpha \delta(g(\alpha)) f(g(\alpha)) \det\left(\frac{\delta g(\alpha)}{\delta \alpha}\right) = \int \mathcal{D}\alpha \delta(\alpha) f(\alpha). \tag{2.26}$$

Choosing $f(\alpha) = 1$, this reduces to

$$\int \mathcal{D}\alpha \delta(g(\alpha)) \det\left(\frac{\delta g(\alpha)}{\delta \alpha}\right) = 1. \tag{2.27}$$

This identity appears promising and indeed, we can insert it into the functional integral to select only physical states. As g must be a function of α we choose $g(\alpha) \equiv G(A^\mu + \partial^\mu \alpha(x)) = \partial_\mu (A^\mu + \partial^\mu \alpha(x)) - \omega(x)$, and insert the left hand side of Eq. (2.27) into the functional integral yielding

$$Z[0] = \int \mathcal{D}A \exp(iS[A]) \left(\int \mathcal{D}\alpha \delta(g(\alpha)) \det\left(\frac{\delta g(\alpha)}{\delta \alpha}\right)\right). \tag{2.28}$$

From the definition of $g(\alpha)$, one can see that $\dfrac{\delta g(\alpha)}{\delta \alpha} = \partial_\mu \partial^\mu$ is independent of α and thus can be factored out as a constant. We then see

$$Z[0] = \det\left(\partial^2\right) \int \mathcal{D}A \exp(iS[A]) \left(\int \mathcal{D}\alpha \delta(G(A^\mu + \partial^\mu \alpha))\right). \tag{2.29}$$

As $S[A_\mu] = S[A_\mu + \partial_\mu \alpha]$ as it is gauge invariant, and $\mathcal{D}A_\mu = \mathcal{D}(A_\mu + \partial_\mu \alpha)$ as the space of functions is similarly invariant, we notice that we can write $Z[0]$ purely in terms of $A_\mu + \partial_\mu \alpha$. However, as it is an integration variable, we can substitute back $A_\mu + \partial_\mu \alpha \to A_\mu$, removing all explicit α dependence to find

$$Z[0] = \det\left(\partial^2\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A \exp\left(iS[A]\right)\delta\left(G(A)\right), \tag{2.30}$$

namely, that the $\mathcal{D}\alpha$ integral factors out, amounting to an infinite constant (without physical implications), and that we have obtained the $\delta(G(A)) = \delta(\partial_\mu A^\mu - \omega(x))$ desired to select only physical states. As this expression holds for any fixed function $\omega(x)$, it holds identically for any normalized linear sum of $\omega(x)$'s. Faddeev and Popov's essential insight was to integrate over a normalized Gaussian weighted envelope in the function space of $\omega(x)$ of standard deviation ξ such that

$$Z[0] = N(\xi)\int \mathcal{D}\omega \exp\left(-i\int \mathrm{d}^4x\, \frac{\omega^2}{2\xi}\right)\int \mathcal{D}A \exp\left(iS[A]\right)\delta\left(\partial_\mu A^\mu - \omega\right), \tag{2.31}$$

where $N(\xi)$ ensures the Gaussian is normalized, and absorbs the other constant factors for brevity. Evaluating the $\mathcal{D}\omega$ integral absorbs the delta function enforcing $\partial_\mu A^\mu = \omega$ and thus yielding

$$Z[0] = N(\xi)\int \mathcal{D}A \exp\left(-i\int \mathrm{d}^4x\, \frac{\left(\partial_\mu A^\mu\right)^2}{2\xi}\right)\exp\left(iS[A]\right), \tag{2.32}$$

where we see that the exponential term modifies the Lagrangian with the addition of a term of the form

$$\mathcal{L}_{\text{gauge fixed}} = \mathcal{L} - \frac{1}{2\xi}\partial_\mu A^\mu \partial_\nu A^\nu. \tag{2.33}$$

where $\xi \in \mathbb{R}$ is referred to as the **gauge parameter** and can be fixed to any desired number. It can be shown that observable quantities will always be independent of your choice of ξ, however some choices more significantly simplify calculations.

This arduous derivation of gauge fixing applies not only to QED, but rather any abelian gauge theory, including ModMax. Namely, given an abelian gauge theory with Lagrangian $\mathcal{L}$, subtracting the gauge fixing term in Eq. (2.33) fixes the gauge, leading to well defined observables and behaviour.

Returning to the momentum space action in Eq. (2.9), we see that after gauge fixing, the action can be written as

$$S[A] = \frac{1}{2}\int \mathrm{d}^4k\, A_\mu(k)\left(-k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k^\mu k^\nu\right)A_\nu(-k), \tag{2.34}$$

which no longer vanishes for $A_\mu = \partial_\mu \alpha$, and thus leads to a well defined generating functional.

2.4 The Effective Action

Further drawing on the analogy with statistical mechanics, recall that the free energy $F(B)$ of a system dependent on a magnetic field B can be obtained from the partition function $Z[B]$ by performing

$$F \equiv -T \ln Z[B]. \tag{2.35}$$

Taking the derivative of F with respect to B then yields the magnetization M of the system,

$$M \equiv -\frac{\partial F}{\partial B}, \tag{2.36}$$

from which the Gibbs free energy G can be found by Legendre transforming F such that

$$G \equiv F - BM. \tag{2.37}$$

Each of these thermodynamic quantities has an analogue within quantum field theory. Namely, the generating functional (which takes the place of the partition function) is defined in terms of an external source $J(x)$, which is the generalization of the external magnetic field B. As such, we can define the analogue of the free energy

$$E[J] \equiv -i\frac{\delta}{\delta J(x)} \ln Z[J], \tag{2.38}$$

for which a further derivative provides the analogue of the magnetization: the expectation value of the field $C_\mu \equiv \langle A_\mu(x)\rangle$ such that

$$\frac{\delta}{\delta J(x)} E[J] = -\langle A_\mu(x)\rangle = -C_\mu. \tag{2.39}$$

Notice that the magnetisation is a global property (i.e. without x_μ dependence) that characterizes the whole system in the same fashion as the expectation value of the field.

Definition 4: Lastly, Legendre transforming $E[J]$, we obtain the **effective action**

$$\Gamma[C] \equiv E[J] - \int \mathrm{d}^4x\, J^\mu(x)\, C_\mu(x), \tag{2.40}$$

which is a functional depending only on C_μ, the expectation value of the field. This action provides an *effective* description of the full theory at a sufficient scale, and thus is an invaluable tool to obtain observable quantities from otherwise unsolvable theories.

Taking the derivative of the effective action with respect to C_μ, it can be shown that

$$\frac{\delta}{\delta C_\mu}\Gamma\left[C_\mu\right] = -J\left(x\right), \tag{2.41}$$

and thus in the sourceless case where $J\left(x\right) = 0$, we find

$$\frac{\delta}{\delta C_\mu}\Gamma\left[C_\mu\right] = -J\left(x\right). \tag{2.42}$$

Namely, $C_\mu = \langle A_\mu\left(x\right)\rangle$ that solves this equation extremizes the action and thus corresponds to a stable solution $A_\mu\left(x\right)$ of the original theory. The effective action therefore allows us to study the large scale *effective* behaviour when quantum effects are cumulatively taken into account.

2.5 Propagators and Correlation Functions

While we do not prove it here, the effective action can equivalently be obtained by evaluating all Feynman diagrams with classical external vertices constructible within a theory. One constructs and evaluates diagrams by obtaining the *Feynman rules* of the theory: namely, the factors to include in the calculation for each possible line and vertex that make up a diagram. As we will see here, each line within a diagram represents the propagation of a particle, and each vertex an interaction.

Definition 5: Given a field $A_\mu\left(x\right)$, the **propagator** of that field satisfies

$$A_\mu\left(x\right) = \int \mathrm{d}^4y\, D_{\mu\nu}\left(x-y\right) A^\nu\left(y\right), \tag{2.43}$$

in that it *propagates* the field $A^\nu\left(y\right)$ to x through all possible paths.

When we draw a Feynman diagram, each internal line represents a propagator corresponding to that field

$$y,\,\nu \sim\!\sim x,\,\mu\; = D_{\mu\nu}\left(x-y\right), \tag{2.44}$$

propagating it from y to x.

With the above gauge fixing, the propagator for the electromagnetic field can now be found to satisfy

$$\left(-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu\right) D_{\nu\rho}\left(k\right) = i\delta^\mu_\rho, \tag{2.45}$$

which has solution

$$D_{\nu\rho}(k) = \frac{-i}{k^2}\left(g_{\nu\rho} - (1-\xi)\frac{k_\nu k_\rho}{k^2}\right). \tag{2.46}$$

Setting $\xi = 1$ here, as we are free to do so without affecting observable quantities, is referred to as *Feynman gauge*, and clearly simplifies the form of the propagator[1] greatly to

$$D_{\nu\rho}(k) = \frac{-ig_{\nu\rho}}{k^2}. \tag{2.47}$$

We will proceed with this choice of gauge for simplicity.

2.6 QED Diagrams

With the propagator for the photon obtained, one can now look at introducing electrons as described in QED. The full Lagrangian is of the form

$$\mathcal{L}_{\text{QED}} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\partial_\mu\gamma^\mu - m\right)\psi}_{\mathcal{L}_0} - e\overline{\psi}\gamma^\mu A_\mu\psi, \tag{2.48}$$

where $\psi(x)$ is a spinor field that describes electrons/positrons, e is the electric charge, and γ^μ are the Dirac matrices that satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

The first two terms in this Lagrangian (together forming $\mathcal{L}_0$) describes how photons and electrons freely evolve, and the third determines their interaction. Like most quantum field theories, this Lagrangian is unsolvable in its exact form and thus we move to using perturbation theory, where we assume the interaction of the electrons and photons is relatively small. This is valid as the electric charge, $e \ll 1$, here determines the strength of this interaction. This allows us to Taylor expand the interaction term such that

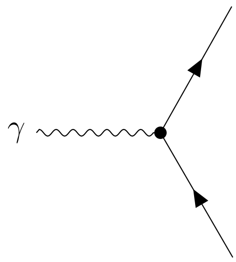
$$\exp\left(i\int \mathrm{d}x\,\mathcal{L}\right) = \exp\left(\mathcal{L}_0\right)\exp\left(i\int \mathrm{d}x\left(-e\overline{\psi}\gamma^\mu A_\mu\psi\right)\right) \tag{2.49}$$

with $\exp x \sim 1 + x + \mathcal{O}\left(x^2\right)$,

$$= \exp\left(\mathcal{L}_0\right)\left(1 - ie\int \mathrm{d}x\,\overline{\psi}\gamma^\mu A_\mu\psi\right) + \mathcal{O}\left(e^2\right), \tag{2.50}$$

[1]Formally, to prevent a pole at $k^2 = 0$, one replaces the denominator with $k^2 + i\varepsilon$ such that the pole is removed, and the limit of $\varepsilon \to 0$ is taken after integration.

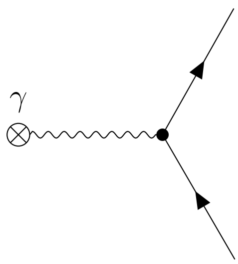
where the 1 term corresponds to the free evolution of particles, and the nontrivial term corresponds to an interaction vertex involving two fermions and a photon. The presence of a $\overline{\psi}$ and a ψ indicates that a fermion enters and leaves this interaction, and the A_μ field indicates that a photon is involved. In fact, with the identification of these fields, the remaining factors in this term yield, in the language of Feynman diagrams, the form of the QED vertex



$$= ie\gamma^\mu \int \mathrm{d}^4 x \,, \tag{2.51}$$

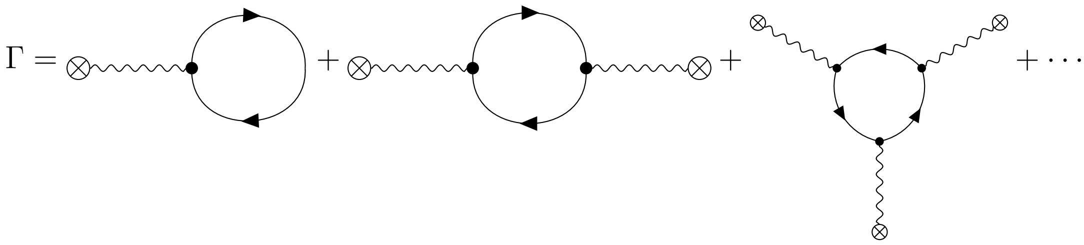
where we draw photons as wavy lines and fermions as straight lines with arrows indicating the flow of charge. This factor is to be included in the evaluation of Feynman diagrams whenever this vertex appears.

If we assume that the photon field is an external classical field $C_\mu(x)$, that is the line exits the diagram, we instead obtain the vertex rule



$$= ie\gamma^\mu \int \mathrm{d}^4 x \, C_\mu(x) \,. \tag{2.52}$$

To obtain the effective action Γ, and thus the form of the largest quantum corrections, we need to construct all possible Feynman diagrams formed out of this vertex that contain one loop. Higher numbers of loops contribute less significantly. Observe that this consists of the following series of diagrams.



(2.53)

Evaluating this series of diagrams using the QED Feynman rules for the vertex and photon propagator (as derived above) yields an effective action, $\Gamma\left[C_\mu\left(x\right)\right]$, that can be used to derive physical observables. This action is referred to as *effective*, as we have integrated out the dependence on electrons/positrons in the evaluation of the Feynman diagrams. Therefore, it is effective in that it describes photons and their self-interaction within the standard model, but is not the full, original theory. This effective action was first derived by Julian Schwinger [14] in 1951. This result was of particular interest as Schwinger also showed that this effective action predicts that at sufficiently high electric field strengths, the vacuum will produce electrons and positrons in pairs. While the field strength required for pair production to occur is unreachable by modern experimentalist techniques, it is widely accepted as a valid prediction of QED.

It is an aim of this project to derive the analogous Feynman rules for ModMax, and to use them to evaluate a similar infinite series of diagrams exactly to find the effective action, $\Gamma\left[C_\mu\left(x\right)\right]$.

3

ModMax in the Background Field Method

With the quantization of QED detailed, we seek to quantize ModMax. However, as well will see, ModMax is not amenable to traditional quantization techniques due to the nonlinear form of the self-interaction. Therefore, I quantize ModMax within the background field method. I show that the one loop effective action for backgrounds with constant field strength exactly vanishes using dimensional regularization as well as the two loop effective action to order γ^2. I further find that allowing the background to vary, logarithmic divergences emerge.

3.1 Background Field Method

Recall that the ModMax Lagrangian [1] is given by

$$\mathcal{L} = S\cosh\gamma + \sqrt{S^2 + P^2}\sinh\gamma, \tag{3.1}$$

where we recall the respectively scalar and pseudoscalar invariants $S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $P = -\frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu}$.

As $\gamma = 0$ recovers Maxwell's Lagrangian, $\mathcal{L} = S$, the most natural interpretation of the ModMax Lagrangian is that the $S\cosh\gamma$ term provides a Maxwell-like free propagation of the photon, and the $\sqrt{S^2 + P^2}\sinh\gamma$ term is an interaction of the photon field with itself. This interaction is small relative to the free evolution, as $\sinh\gamma \overset{\mathcal{O}(\gamma)}{\sim} \gamma \ll 1 \overset{\mathcal{O}(\gamma)}{\sim} \cosh\gamma$.

Thus, as this is a self-interacting theory, one may desire to proceed using canonical quantization techniques as applied to QED. However, the nonlinear form of the interaction is not compatible with the familiar perturbative Feynman diagram expansion, where we require positive integer powers of the fields to proceed. The natural impulse is then to Taylor expand the interaction, assuming that the essential physics can be captured at low powers of the field, or equivalently by weak fields (as higher powers would be comparatively negligible). However, ModMax does not reduce to Maxwell's equations in the weak field limit, only in the $\gamma \to 0$ limit, and thus such an approach is ill-suited. Clearly an alternative approach is needed.

As such, we begin by employing the *background field method* in which we consider a fixed classical background with quantum fluctuations about this background [15] (See Fig. 3.1). Mathematically, this is performed by taking the photon field A_μ and decomposing it into a classical field C_μ and a quantum field a_μ such that the quantum oscillations on the classical background are equivalent to the original field with

$$A_\mu = C_\mu + a_\mu. \tag{3.2}$$

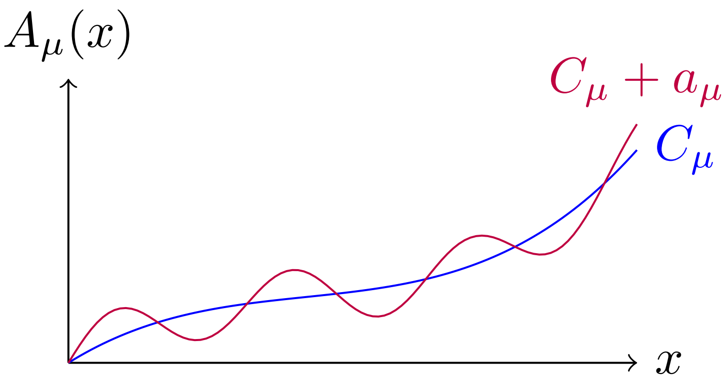


Figure 3.1: Pictorial depiction of background field method splitting of an arbitrary field $A_\mu(x)$ into a classical background field and a quantum oscillation about the background. The classical field here is varying spatially, but can also be taken to be constant.

This linear splitting of the field leads to a linear splitting in the field strength of

$$\Rightarrow F_{\mu\nu} = \partial_\mu (C_\nu + a_\nu) - \partial_\nu (C_\mu + a_\mu) \tag{3.3}$$

into classical and quantum field strengths given by

$$= \underbrace{(\partial_\mu C_\nu - \partial_\nu C_\mu)}_{C_{\mu\nu}} + \underbrace{(\partial_\mu a_\nu - \partial_\nu a_\mu)}_{f_{\mu\nu}} \tag{3.4}$$

$$\equiv C_{\mu\nu} + f_{\mu\nu}, \tag{3.5}$$

where $C_{\mu\nu}$ is the field strength tensor for the classical field C_μ and $f_{\mu\nu}$ is the field strength tensor for the quantum field a_μ. This splitting now allows us to tackle the nonlinear terms in the Lagrangian perturbatively, as we can expand in powers of the quantum terms around a fixed classical background field term.

Additionally, to simplify the problem further, we can assume that the classical field is stationary both in space and time. This reduces the predictive power of the calculations, but simplifies the process dramatically. This assumption is equivalent to neglecting $\partial_\mu C_{\nu\rho}$ as negligible in the effective action, $\Gamma[C_\mu]$. Schwinger [14] made this assumption when calculating the effective action for QED. We will first investigate the case where we consider the background stationary before returning to generalize to the more difficult case.

3.2 Taylor Expansion

We notice that the invariants S and P can thus be decomposed into background and quantum field strength tensors as

$$S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \underbrace{-\frac{1}{4}C_{\mu\nu}C^{\mu\nu}}_{S_C} - \frac{1}{2}C_{\mu\nu}f^{\mu\nu} \underbrace{-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}}_{S_a} \tag{3.6}$$

$$S = S_C - \frac{1}{2}C_{\mu\nu}f^{\mu\nu} + S_a \tag{3.7}$$

where we desire to make use of the constant classical field strength $\partial_\mu C_{\nu\rho} = 0$. This in fact implies that the cross term vanishes as

$$C_{\mu\nu}(\partial^\mu a^\nu) = \partial^\mu(C_{\mu\nu}a^\nu) - (\partial^\mu C_{\mu\nu})\,a^\nu \tag{3.8}$$

where the first term is a total derivative that amounts to a boundary term in the action (that does not influence the physics) and the second is a derivative of the classical field strength. Therefore, we have $C_{\mu\nu}f^{\mu\nu} = 0$ and can write

$$S = S_C + S_a \tag{3.9}$$

and identically we have

$$P = P_C + P_a, \tag{3.10}$$

where $P_C \equiv -\frac{1}{4}C_{\mu\nu}\widetilde{C}^{\mu\nu}$ and $P_a \equiv -\frac{1}{4}f_{\mu\nu}\widetilde{f}^{\mu\nu}$ analogously.

Note that the cross terms do not vanish in S^2 and P^2 as they are no longer a total derivative and we instead have

$$S^2 = S_C^2 - \underbrace{S_C C_{\mu\nu} f^{\mu\nu}}_{\text{topological}} + 2S_C S_a + \frac{1}{4}C_{\mu\nu}C_{\rho\tau}f^{\mu\nu}f^{\rho\tau} - \underbrace{S_a C_{\mu\nu} f^{\mu\nu}}_{\mathcal{O}(a^3)} + \underbrace{S_a^2}_{\mathcal{O}(a^4)} . \tag{3.11}$$

We refer to total derivative terms as *topological*, as they only contribute in nontrivial topological spaces beyond our study. Neglecting terms of order $\mathcal{O}\left(a^3\right)$ and greater, we are left with

$$S^2 = S_C^2 - \underbrace{S_C C_{\mu\nu} f^{\mu\nu}}_{\text{topological}} + 2S_C S_a + \frac{1}{4}C_{\mu\nu}C_{\rho\tau}f^{\mu\nu}f^{\rho\tau} \tag{3.12}$$

$$P^2 = P_C^2 - \underbrace{P_C \widetilde{C}_{\mu\nu} f^{\mu\nu} + 2P_C P_a}_{\text{topological}} + \frac{1}{4}\widetilde{C}_{\mu\nu}\widetilde{C}_{\rho\tau}f^{\mu\nu}f^{\rho\tau} \tag{3.13}$$

where discarding topological terms, we arrive at

$$\Rightarrow S^2 + P^2 = S_C^2 + P_C^2 + 2S_C S_a + \frac{1}{4}\left(C_{\mu\nu}C_{\rho\tau} + \widetilde{C}_{\mu\nu}\widetilde{C}_{\rho\tau}\right)f^{\mu\nu}f^{\rho\tau}. \tag{3.14}$$

We see that the first two terms, $S_C^2 + P_C^2$, are purely background dependent and thus serve as a nontrivial classical point to Taylor expand with respect to. Namely, Taylor expanding $\sqrt{S^2+P^2}$ about $S_C^2 + P_C^2$ we see that

$$\sqrt{S^2+P^2} \equiv \sqrt{S_C^2 + P_C^2 + Q} = \sqrt{S_C^2 + P_C^2} + \frac{Q}{2\sqrt{S_C^2 + P_C^2}} - \frac{Q^2}{8\left(S_C^2 + P_C^2\right)^{\frac{3}{2}}} + \mathcal{O}\left(Q^3\right) \tag{3.15}$$

where the quantum terms are grouped with

$$Q \equiv 2S_C S_a + \underbrace{2P_C P_a - \left(S_C C_{\mu\nu} + P_C \widetilde{C}_{\mu\nu}\right)f^{\mu\nu}}_{\text{topological}} + \frac{1}{4}\left(C_{\mu\nu}C_{\rho\tau} + \widetilde{C}_{\mu\nu}\widetilde{C}_{\rho\tau}\right)f^{\mu\nu}f^{\rho\tau} \tag{3.16}$$

where the topological terms are included as they contribute in powers of Q, and up to terms quadratic in the quantum field we have

$$Q^2 = \left(S_C^2 C_{\mu\nu}C_{\rho\tau} + S_C P_C\left(\widetilde{C}_{\mu\nu}C_{\rho\tau} + C_{\mu\nu}\widetilde{C}_{\rho\tau}\right) + P_C^2\widetilde{C}_{\mu\nu}\widetilde{C}_{\rho\tau}\right)f^{\mu\nu}f^{\rho\tau} + \mathcal{O}\left(a^3\right). \tag{3.17}$$

Therefore with the classical field Lagrangian defined by

$$\mathcal{L}_C \equiv S_C \cosh\gamma + \sqrt{S_C^2 + P_C^2}\sinh\gamma \tag{3.18}$$

we can thus write the full Lagrangian as

$$\mathcal{L} = \mathcal{L}_C + S_a \cosh\gamma + \left(2S_C S_a + B_{\mu\nu\rho\tau} f^{\mu\nu} f^{\rho\tau}\right)\sinh\gamma + \mathcal{O}\left(a^3\right), \tag{3.19}$$

where I further define

$$B_{\mu\nu\rho\tau} \equiv \frac{C_{\mu\nu}C_{\rho\tau} + \widetilde{C}_{\mu\nu}\widetilde{C}_{\rho\tau}}{8\sqrt{S_C^2 + P_C^2}} - \frac{S_C^2 C_{\mu\nu}C_{\rho\tau} + S_C P_C\left(\widetilde{C}_{\mu\nu}C_{\rho\tau} + C_{\mu\nu}\widetilde{C}_{\rho\tau}\right) + P_C^2\widetilde{C}_{\mu\nu}\widetilde{C}_{\rho\tau}}{8\left(S_C^2 + P_C^2\right)^{\frac{3}{2}}}, \tag{3.20}$$

to capture the classical field dependence in the interaction. This tensor coincidentally has the same symmetries as the Riemann curvature tensor $B_{\mu\nu\rho\tau} = B_{\rho\tau\mu\nu}$ and $B_{\mu\nu\rho\tau} = -B_{\nu\mu\rho\tau} = -B_{\mu\nu\tau\rho}$. Here it is constant as it is purely a function of the classical field strength and its dual.

The Lagrangian for the quantum field thus suggests the quantum field has a Maxwell-like propagation $S_a \cosh\gamma$ and an interaction vertex quadratic in the quantum field a_μ.

As with the QED Lagrangian, we have that we can rearrange S_a into a more useful form with

$$S_a = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} = \frac{1}{2} a_\nu \left(\partial_\rho\partial^\rho g^{\mu\nu} - \partial^\mu\partial^\nu\right) a_\mu \tag{3.21}$$

and using the symmetry of $B_{\mu\nu\rho\tau}$, we have

$$B^{\mu\nu\rho\tau} f_{\mu\nu} f_{\rho\tau} = 4a_\nu \left(B^{\mu\nu\rho\tau}\partial_\mu\partial_\rho\right) a_\tau, \tag{3.22}$$

the Lagrangian can be expressed as

$$\begin{aligned}\mathcal{L} = \mathcal{L}_C &+ \frac{\cosh(\gamma)}{2} a_\nu \left(\partial_\rho\partial^\rho g^{\mu\nu} - \partial^\mu\partial^\nu\right) a_\mu \\ &+ \sinh(\gamma)\, a_\nu \left(S_C\partial_\rho\partial^\rho g^{\mu\nu} - S_C\partial^\mu\partial^\nu a_\mu + 4B^{\alpha\nu\beta\mu}\partial_\alpha\partial_\beta\right) a_\mu.\end{aligned} \tag{3.23}$$

As ModMax is an abelian gauge theory, it must be gauge fixed for this Lagrangian to give meaningful results. Applying the Faddeev-Popov procedure results in the addition of the gauge fixing term $-\xi a_\nu \partial^\nu \partial^\mu a_\mu$ where ξ here can be background dependent. Taking the Feynman gauge equivalent of $\xi = \frac{1}{2}\cosh\gamma - S_C \sinh\gamma$ the Lagrangian simplifies to

$$\mathcal{L} = \mathcal{L}_C + \frac{\cosh(\gamma)}{2} a_\nu \partial_\rho \partial^\rho g^{\mu\nu} a_\mu + \sinh(\gamma)\, a_\nu \left(S_C \partial_\rho \partial^\rho g^{\mu\nu} + 4B^{\alpha\nu\beta\mu}\partial_\alpha \partial_\beta\right) a_\mu. \tag{3.24}$$

With $\partial_\mu \to ik_\mu$, this Lagrangian can be expressed in momentum space as

$$\mathcal{L} = \mathcal{L}_C - \frac{\cosh(\gamma)}{2} a_\nu k^2 g^{\mu\nu} a_\mu - \sinh(\gamma)\, a_\nu \left(S_C k^2 g^{\mu\nu} + 4B^{\alpha\nu\beta\mu} k_\alpha k_\beta\right) a_\mu. \tag{3.25}$$

Note. We consider the last term in this Lagrangian as an interaction term despite the classical field strength tensors that appear being non-dynamical. Namely, as we have $\partial_\mu C_{\nu\rho} = 0$, the classical field strengths are independent of x_μ and thus can be factored out of the action integral as constants. However this term in the Lagrangian still represents an interaction of the quantum field with a fixed background source distinct from the propagator term.

With the $\sinh\gamma$ term interpreted as an interaction vertex between the classical and quantum fields, the momentum space propagator for the quantum field is then given by

$$-\cosh\gamma k^2 g_{\mu\nu} D^{\nu\rho} = i\delta^\rho_\mu \tag{3.26}$$

$$\Rightarrow D^{\nu\rho} = \frac{1}{\cosh\gamma}\frac{-ig^{\nu\rho}}{k^2}, \tag{3.27}$$

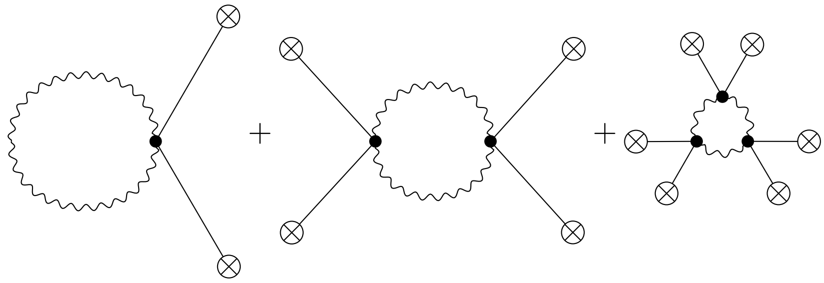
which is entirely analogous to the propagator for the quantum field in QED. See Appendix D for a derivation of this propagator.

3.3 One Loop Effective Action

I will draw quantum fields as wavy lines and the background field as an unlabelled solid line (when not omitted).

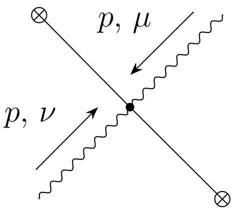
To obtain the one loop effective action Γ, it remains to evaluate all Feynman diagrams containing at most one loop that are constructible out of the quantum field propagator and the interaction vertex. This corresponds to a single infinite series of diagrams given by

$\Gamma =$



$+ \cdots . \quad (3.28)$

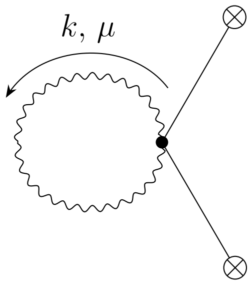
To evaluate such diagrams, we need to derive the vertex factor for ModMax, which together with the propagator is referred to as the *Feynman rules*. Reading off the Lagrangian, we see that the interaction vertex takes the form



$$= -2\sinh(\gamma)\left(S_C p^2 g^{\mu\nu} + 4B^{\alpha\nu\beta\mu} p_\alpha p_\beta\right), \tag{3.29}$$

where notice that no momentum can flow through the classical fields in the interaction as $\partial_\mu C_{\nu\rho} = 0$.

The first diagram in this infinite series is given by



$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \sinh(\gamma)\left(2S_C k^2 g^{\mu\nu} + 8B^{\alpha\nu\beta\mu} k_\alpha k_\beta\right) D_{\nu\mu} \tag{3.30}$$

where the replacement $k_\alpha k_\beta \to \frac{k^2}{4} g_{\alpha\beta}$ and the propagator derived above yield

$$= -i\coth(\gamma) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(2S_C k^2 g^{\mu\nu} + 8B^{\alpha\nu\beta\mu} g_{\alpha\beta} k^2\right) \frac{g_{\nu\mu}}{k^2} \tag{3.31}$$

$$= -8i\coth(\gamma)\left(S_C + B^{\alpha\nu}{}_{\alpha\nu}\right) \int \frac{\mathrm{d}^d k}{(2\pi)^d} 1. \tag{3.32}$$

Notice the lack of k dependence in this integral. Proceeding regardless, this contraction simplifies greatly with the use of the identity

$$-\frac{1}{4}\widetilde{C}_{\mu\nu}\widetilde{C}^{\mu\nu} = -S_C \tag{3.33}$$

to

$$B^{\alpha\nu}{}_{\alpha\nu} = -\frac{S_C^3 + S_C P_C^2 - S_C P_C^2}{2\left(S_C^2 + P_C^2\right)^{\frac{3}{2}}} \tag{3.34}$$

$$B^{\alpha\nu}{}_{\alpha\nu} = -\frac{S_C^3}{2\left(S_C^2 + P_C^2\right)^{\frac{3}{2}}}. \tag{3.35}$$

However, as the integral is independent of k, it is scaleless and thus while Λ^d divergent with a naive cutoff, using dimensional regularization we can show it is exactly zero.

3.4 Dimensional Regularization

Divergent quantities that one would naively expect to be physical are a common occurrence in quantum field theory. Any sufficiently sophisticated theory gives rise to such divergences at some number of loops, and requires *renormalization*, the process of recovering finite physical results from such theories by redefining constants in the Lagrangian [8, 16, 17]. Such redefinitions absorb the divergences that arise.

However, first one must characterize the divergence, the process of which is referred to as *regularization*. This process is not unique and there are many *regulators* one can make use of. For example, for the above integral, the naive method is called *cutoff regularization* where one introduces a maximum momentum scale $k^2 \leq \Lambda^2$ such that our integral now reads,

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} 1 \longrightarrow \int_{-\Lambda}^{\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} 1. \tag{3.36}$$

One would then have Λ dependence in the Lagrangian, with the intention of taking the limit of $\Lambda \to \infty$ to recover the original theory.

Note. The final result of regularization is expected to be independent of the regulator used. If two regulators lead to different observable quantities, often a symmetry is being broken by one or both regulators.

While cutoff regularization is approachable, it is not the most elegant method. This is largely caused by the introduction of a characteristic length scale Λ in our otherwise scaleless theory.

Instead, we will make use of *dimensional regularization* where we consider the dimension to be a free parameter $d \in \mathbb{R} \setminus \mathbb{N}$. The natural numbers $\mathbb{N}$ are excluded from the domain here as they lead to divergences and thus undefined values. However, the limit in which $d \to \mathbb{N}$ (usually $d = 4$) is well defined. We then appeal to *analytic continuation*, where as a given calculation is an analytic function of the dimension d, we define the value at $d = 4$ to agree with the limit. This is a valid regulator, and introduces no characteristic scale. We use dimensional regularization throughout this thesis as it causes a large number of otherwise difficult integrals to vanish.

Claim. In fact, returning to the integral at hand, using dimensional regularization, we can show that

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} 1 \to 0, \tag{3.37}$$

as $d \to \mathbb{N}$.

Proof. We begin by considering $d \notin \mathbb{Z}$ dimensional Euclidean space (rather than Minkowski) for convenience. Taking the integral

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} 1 = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{k^2+m^2} + \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{m^2}{k^2+m^2}. \tag{3.38}$$

This is a specific case of a known integral [16], with general form given by

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{2\beta}}{(k^2+m^2)^\alpha} = \frac{\Gamma\left(\beta+\frac{d}{2}\right)\Gamma\left(\alpha-\beta-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma(\alpha)\Gamma\left(\frac{d}{2}\right)} m^{2\left(\frac{d}{2}-\alpha+\beta\right)}, \tag{3.39}$$

where $\Gamma(n+1) = n!$ is the *Gamma function*, the generalization of the factorial to the real numbers using analytic continuation. With $\alpha = \beta = 1$ we see that

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{k^2+m^2} = \frac{\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} m^d \tag{3.40}$$

where $\Gamma(x+1) = x\Gamma(x)$ implies

$$= \left(\frac{d}{2}\right) \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} m^d \tag{3.41}$$

and for $\beta = 0$ we find

$$m^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} = m^2 \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} m^{d-2} \tag{3.42}$$

where $\Gamma(x+1) = x\Gamma(x)$ similarly implies

$$= \left(-\frac{d}{2}\right) \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} m^d, \tag{3.43}$$

which is the negative of the previous integral, and thus their sum vanishes as desired $\forall d \in \mathbb{R} \setminus \mathbb{N}$. By appealing to analytic continuation of this result, it holds identically for $d = 4$. □

In fact, the above argument is generalizable to show that any integral of the form

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} k^{2\alpha}, \tag{3.44}$$

vanishes in dimensional regularization for $\alpha \in \mathbb{Z}$. Intuitively, this is because the integral has no characteristic external scale dependence which is central in dimensional regularization [18].

3.5 Generalization to nth order diagrams

Notice that the above result relies only on the momentum dependence of the integral.

For a general nth order one loop diagram however, all insertions of additional vertices do not change the momentum dependence of the integral. Namely, as we will have n propagators (Eq. (3.27)) $D_{\mu\nu} \propto \frac{1}{k^2}$ and n vertices (Eq. (3.29)) $\propto k^2$ which are all equal by momentum conservation (as we assumed no momentum flow through the classical fields), the integral will be momentum independent as for the 1 vertex diagram.

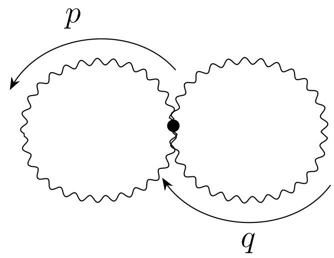
In fact, as the vertex factor contains $k_\alpha k_\beta$ rather than k^2 we will first have to perform the symmetrization of n metric tensors that leads to the replacement [8]

$$\prod_{i=1}^{n} k^{\alpha_{2i}} k^{\alpha_{2i+1}} \to \frac{k^{2n}(d-2)!!}{2^{\frac{n}{2}}(n-2+d)!!} g^{(\alpha_1\alpha_2} \cdots g^{\alpha_{2n}\alpha_{2n+1})}. \tag{3.45}$$

Therefore we conclude that the perturbative one-loop effective action Γ, with constant background field strength $C_{\mu\nu}$, vanishes through dimensional regularization. This implies that there are no 1-loop corrections to the classical theory under these assumptions.

3.6 Two Loops

The only two loop diagram to order γ (i.e. having one vertex as each carries $\sinh\gamma \overset{\mathcal{O}(\gamma)}{=} \gamma$) is quartic in the quantum field



This vertex and diagram is also quartic in the classical field but these lines have been suppressed for clarity (and due to the fact that they carry no momenta). To evaluate this however, we need to return to the expansion of the square root

$$\begin{aligned}\sqrt{S_C^2+P_C^2+Q} &= \sqrt{S_C^2+P_C^2} + \frac{Q}{2\sqrt{S_C^2+P_C^2}} - \frac{Q^2}{8\left(S_C^2+P_C^2\right)^{\frac{3}{2}}} \\ &+ \frac{Q^3}{16\left(S_C^2+P_C^2\right)^{\frac{5}{2}}} - \frac{5Q^4}{128\left(S_C^2+P_C^2\right)^{\frac{7}{2}}}. \end{aligned} \tag{3.46}$$

From Q and a_μ power counting, we have that $\mathcal{O}\left(a^4\right)$ term in the Lagrangian is

$$\begin{aligned}\frac{\mathcal{L}_{a^4}}{\sinh\gamma} &= \frac{1}{2\sqrt{S_C^2+P_C^2}} S_a^2 \\ &- \frac{1}{8\left(S_C^2+P_C^2\right)^{\frac{3}{2}}}\left[\left(2S_CS_a + 2P_CP_a + \frac{1}{4}\left(C_{\mu\nu}C_{\rho\tau} + \widetilde{C}_{\mu\nu}\widetilde{C}_{\rho\tau}\right) f^{\mu\nu} f^{\rho\tau}\right)^2 \right.\\ &\left. + 2\left(\left(S_CC_{\mu\nu} + P_C\widetilde{C}_{\mu\nu}\right) f^{\mu\nu}\right)\left(\left(S_aC_{\alpha\beta} + P_a\widetilde{C}_{\alpha\beta}\right) f^{\alpha\beta}\right)\right] \\ &+ \frac{3}{16\left(S_C^2+P_C^2\right)^{\frac{5}{2}}}\left(\left(S_CC_{\alpha\beta} + P_C\widetilde{C}_{\alpha\beta}\right) f^{\alpha\beta}\right)^2 \\ &\times\left(2S_CS_a + 2P_CP_a + \frac{1}{4}\left(C_{\mu\nu}C_{\rho\tau} + \widetilde{C}_{\mu\nu}\widetilde{C}_{\rho\tau}\right) f^{\mu\nu} f^{\rho\tau}\right)\end{aligned}$$

$$-\frac{5}{128\left(S_C^2+P_C^2\right)^{\frac{7}{2}}}\left(\left(S_C C_{\mu\nu}+P_C\widetilde{C}_{\mu\nu}\right)f^{\mu\nu}\right)^4. \tag{3.47}$$

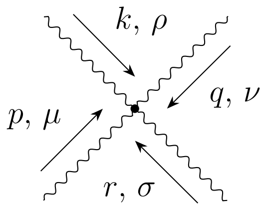
This is unapproachable to ascertain the exact form of the background dependence. However, as the quantum fields only appear in the field strengths, we notice that we can write the vertex

$$\mathcal{L}_{Q^4}=\sinh\left(\gamma\right)A_{\mu\nu\rho\tau\alpha\beta\sigma\kappa}f^{\mu\nu}f^{\rho\tau}f^{\alpha\beta}f^{\sigma\kappa} \tag{3.48}$$

where A is a background dependent, momentum independent tensor that captures the structure of the vertex. It's exact form is not important as we will see. Denoting anti-symmetrization with $A_{[\mu\nu]}=A_{\mu\nu}-A_{\nu\mu}$ observe that we can simplify this to

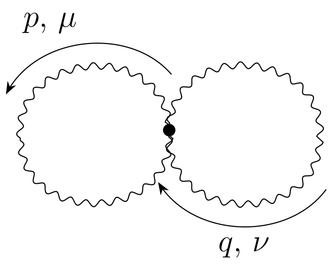
$$\mathcal{L}_{Q^4}=\sinh\left(\gamma\right)A_{[\mu\nu][\rho\tau][\alpha\beta][\sigma\kappa]}\partial^\mu a^\nu\partial^\rho a^\tau\partial^\alpha a^\beta\partial^\sigma a^\kappa. \tag{3.49}$$

As each field has a derivative acting on it, we can express this vertex as



$$=\sinh\left(\gamma\right)B^{\alpha\ \beta\ \kappa\ \tau}_{\ \mu\ \nu\ \rho\ \sigma}p_\alpha q_\beta k_\kappa r_\tau, \tag{3.50}$$

where $B^{\alpha\ \beta\ \kappa\ \tau}_{\ \mu\ \nu\ \rho\ \sigma}$ is also independent of the momenta and captures both the background dependence (from the A tensor) and the symmetrization of momenta from different contractions. The explicit form of B is also not important. Therefore the double loop diagram can be written with symmetry factor $S=8$ [8] as



$$=\frac{\sinh\gamma}{8}B^{\alpha\ \beta\ \kappa\ \tau}_{\ \mu\ \nu\ \rho\ \sigma}\int\frac{\mathrm{d}^dp\,\mathrm{d}^dq}{(2\pi)^{2d}}p_\alpha p_\beta q_\kappa q_\tau D^{\mu\rho}D^{\nu\sigma} \tag{3.51}$$

where with the propagator from Eq. (3.27) $D^{\mu\rho}=\frac{1}{\cosh\gamma}\frac{-ig^{\mu\rho}}{k^2}$ we have

$$=\frac{\coth\gamma}{8\cosh\gamma}B^{\alpha\ \beta\ \kappa\ \tau}_{\ \mu\ \nu\ \rho\ \sigma}\int\frac{\mathrm{d}^dp\,\mathrm{d}^dq}{(2\pi)^{2d}}p_\alpha p_\beta q_\kappa q_\tau\frac{g^{\mu\rho}g^{\nu\sigma}}{p^2q^2} \tag{3.52}$$

where applying the metric tensors yields

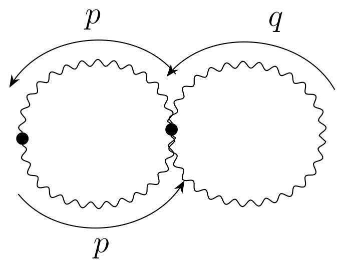
$$= \frac{\coth\gamma}{8\cosh\gamma} B^{\alpha\ \ \beta\ \ \kappa\mu\tau\nu}_{\ \ \mu\ \ \nu} \int \frac{\mathrm{d}^d p\,\mathrm{d}^d q}{(2\pi)^{2d}} \frac{p_\alpha p_\beta q_\kappa q_\tau}{p^2 q^2} \tag{3.53}$$

where as before symmetry implies we can take $p_\alpha p_\beta \to \frac{p^2}{4} g_{\alpha\beta}$ and identically for q yielding

$$= \frac{\coth\gamma}{128\cosh\gamma} B_{\beta\mu\ \ \nu\tau}^{\ \ \ \ \beta\ \ \ \ \mu\tau\nu} \int \frac{\mathrm{d}^d p\,\mathrm{d}^d q}{(2\pi)^{2d}} 1, \tag{3.54}$$

for which, as we found above, a naive cutoff would suggest Λ^{2d} divergence, however the dimensional regularization result derived above allows us to conclude it vanishes.

Notice that we can insert a vertex along either of these loops which will not change the momentum structure of the diagram and get us to γ^2 order



,

and thus we conclude that this diagram will also vanish by similar arguments.

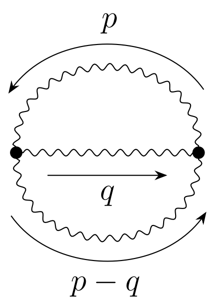
The only other diagram arising at order γ^2 has two cubic vertices, which will come from a Lagrangian

$$\mathcal{L}_{a^3} = A_{[\mu\nu][\rho\tau][\alpha\beta]} \partial_\mu a_\nu \partial_\rho a_\tau \partial_\alpha a_\beta, \tag{3.55}$$

and leads to a vertex of the form

$$\sinh(\gamma)\, B^{\mu\ \ \rho\ \ \alpha}_{\ \ \nu\ \ \tau\ \ \beta} k^1_\mu k^2_\rho k^3_\alpha. \tag{3.56}$$

This diagram does not vanish and the full calculation using dimensional regularization is shown in Appendix A. The diagram evaluates to



$$= (-i)^3 \frac{\sinh^2\gamma}{2\cosh^3\gamma} B^{\mu_1\ \ \rho_1\ \ \alpha_1}_{\ \ \nu_1\ \ \tau_1\ \ \beta_1} B^{\mu_2\nu_1\rho_2\tau_1\alpha_2\beta_1} \times$$

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q_{\rho_1} q_{\rho_2} q_{\alpha_1} q_{\alpha_2}}{q^2} \frac{i\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}(d-1)} \left[q_{\mu_1} q_{\mu_2} q^{d-4} \left(1-\frac{d}{2}\right) - \frac{d g_{\mu_1\mu_2}}{8} q^{d-2} \right]. \tag{3.57}$$

Ignoring the prefactors, we take $d = 4 + 2\varepsilon$ with the intention of taking the limit $\varepsilon \to 0 \Rightarrow d \to 4$. Using the expansion of

$$\Gamma(-1-\varepsilon) = \frac{1}{\varepsilon} - \gamma + 1, \tag{3.58}$$

we find that the divergent part of this integral is

$$\frac{1}{\varepsilon} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{q_{\rho_1} q_{\rho_2} q_{\alpha_1} q_{\alpha_2}}{q^2} \frac{i}{(4\pi)^2 6} \left[2 q_{\mu_1} q_{\mu_2} + g_{\mu_1\mu_2} q^2 \right]. \tag{3.59}$$

This term is *logarithmically divergent*, and similarly to before, due to the presence of the prefactor tensors, does not resemble the original Lagrangian. Notice that where $\forall \varepsilon > 0 \Rightarrow d \neq 4$, we are left with a symmetrizable integral over q that will vanish identically. By analytic continuation, in this regularization scheme we therefore conclude that the integrals also vanish at $d = 4$.

Therefore the two loop effective action also vanishes at minimum to order γ^2. Proceeding further in this manner is impractical but we expect that a generalization can be made to suggest that the effective action should vanish at all orders in γ and at all loops. Allowing derivatives to act on the field strength such that $\partial_\mu C_{\nu\rho} \neq 0$ contrary to what was assumed here, will lead to logarithmic divergences and even constant diagrams as we will see below that will not vanish in dimensional regularization.

3.7 Varying Backgrounds

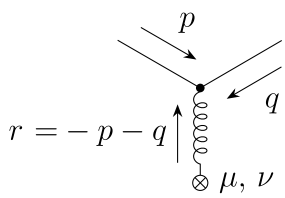
Proceeding in a similar manner to before, we perform the background field method splitting, only now without discarding terms of the form $\partial_\mu C_{\nu\rho}$ (as well as higher derivatives). This leads to an entirely analogous Lagrangian

$$\mathcal{L} = \mathcal{L}_C + S\cosh\gamma + \sinh\gamma a_\mu P^{\mu\nu} a_\nu \tag{3.60}$$

where the background field dependence within $P^{\mu\nu}$ has become more complex with

$$P^{\mu\nu} = \left(S_C \partial^\rho + \partial^\rho S_C\right) \partial_\rho g^{\mu\nu} - \partial^\nu S_C \partial^\mu - \partial_\alpha P_C \varepsilon^{\alpha\nu\rho\mu} \partial_\rho + 2B^{\tau\mu\rho\nu} \partial_\tau \partial_\rho + 2\partial_\tau B^{\tau\mu\rho\nu} \partial_\rho. \tag{3.61}$$

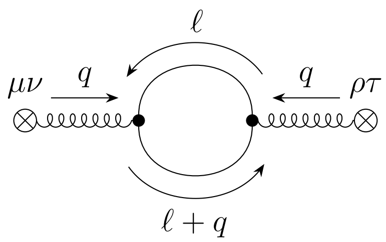
When we held $\partial_\mu C_{\nu\rho} = 0$ we were able to factor the classical field dependence out of the integral as it was necessarily independent of x^μ. However in the general varying background case, this is no longer possible. Nonetheless, if we consider $P^{\mu\nu}(x)$ to represent the cumulative effect of the background rather than just a composite operator, then we can obtain Feynman rules for this theory. Namely, we see in the interaction vertex (with the prefactor $\sinh\gamma$), that we have two factors of the quantum photon field a_μ and one $P_{\mu\nu}$. As such, our interaction vertex has two quantum photons (represented by wavy lines) and one cumulative classical background photon (represented by a coiled line)



$$= -i\sinh\gamma \int \mathrm{d}^4x\, P_{\mu\nu}(x)\,, \tag{3.62}$$

entirely analogously to the QED case.

The only one-vertex diagram constructible in from this vertex vanishes by an identical argument as in the constant background case, thus we move to the one loop two vertex diagram. Applying the derived Feynman rules (i.e. the vertex factor and propagator) we see that this diagram yields



$$= \frac{\sinh^2\gamma}{\cosh^2\gamma}\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{g^{\mu\rho}g^{\nu\tau}}{\ell^2\left(\ell^2+q^2\right)} \tag{3.63}$$

Applying dimensional regularization, we find by a similar argument as above,

$$= \frac{1}{d}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}q^{d-4} \tag{3.64}$$

where adding back in the prefactor external classical field dependence, we can quote the final result as

$$= \frac{1}{d-3}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int\frac{\mathrm{d}^dq}{(2\pi)^d}P_{\mu\nu}(q)\,q^{d-4}P^{\mu\nu}(-q) \tag{3.65}$$

where using $\Gamma(1+x) = x\Gamma(x)$,

$$= -\frac{d}{d-3}\left(1-\frac{d}{2}\right)\frac{\Gamma\left(-\frac{d}{2}\right)}{2\left(4\pi\right)^{\frac{d}{2}}}\int \frac{\mathrm{d}^d q}{\left(2\pi\right)^d} P_{\mu\nu}\left(q\right) q^{d-4} P^{\mu\nu}\left(-q\right) \tag{3.66}$$

With $d = 4 + 2\varepsilon$ and taking $\varepsilon \to 0$ such that $d \to 4$, we see

$$= \left(\frac{1}{\varepsilon}\right)\frac{2}{\left(4\pi\right)^2}\int \frac{\mathrm{d}^4 q}{\left(2\pi\right)^4} P_{\mu\nu}\left(q\right) P^{\mu\nu}\left(-q\right), \tag{3.67}$$

which is not only divergent as $\varepsilon \to 0$, but is also not of the form of the original Lagrangian. If this result was of the form of the original Lagrangian, then we can interpret such a diagram as suggesting a redefinition of a constant (such as γ) within the original Lagrangian, to absorb this divergence. However, as this is not the case, this suggests that ModMax is likely not a physical theory on the quantum domain, without modification. Note that it is possible that there is a structure hidden in such divergences which is of the form of the original Lagrangian, which may only appear when evaluated to all loops or all orders in γ. As this is infeasible to ascertain directly due to the increased complexity with order, we move to study ModMax's two dimensional analogue which may elucidate further insight.

Nonetheless, we have obtained the effective action for ModMax in both the static and varying background case, thus characterizing the predictions of the theory on the quantum level.

4

Scalar Field Analogue in Two Dimensions

With the quantization of ModMax characterized through the two-loop effective action, I similarly study the scalar analogue of ModMax in $d = 2$ and show that the methods applied to ModMax generalized appropriately. Two dimensional conformal field theory is a highly active area of research, as conformal symmetry is extremely restrictive and thus powerful in $1 + 1$ spacetime dimensions. We expect nontrivial results to emerge at a lower number of loops and thus such a toy model can provide insight into the underlying physics.

Similarly, upon successfully quantizing a conformal field theory, the corrections to the classical theory that arise can lead to the discovery of novel conformal field theories at the classical level. This phenomenon can be referred to as a classical conformal field theory *generated* quantum mechanically. However, if the conformal symmetry is broken at the quantum level, referred to as an *anomaly*, this is equally of interest as this often leads to observable predictions of the theory. The corrections I obtain appear to respect conformal symmetry, however further investigation is required into the properties of such generated classical conformal field theories.

Through the generalization of the approach I developed above, I find that the one loop effective action for backgrounds with constant field strength identically vanishes as in the original theory. By allowing the background to vary, I find that the first corrections to the classical theory arise at the order $\mathcal{O}\left(\gamma^2\right)$, and resemble a

QED-like operator dependent on the classical field. Motivated by the experimental bound of $\gamma \leq 3 \times 10^{-22}$, I further calculated the contribution from all diagrams up to order $\mathcal{O}(\gamma^2)$, namely the infinite series of diagrams containing up to 2 vertices.

4.1 Background Field Method

Recall that in $d = 4$, the ModMax Lagrangian [1, 7] is given by

$$\mathcal{L} = S \cosh\gamma + \sqrt{S^2 + P^2} \sinh\gamma. \tag{4.1}$$

By dimensional reduction from ModMax in $d = 4$, one can define in $d = 2$ for $N \geq 2$ scalar boson fields φ^i, (where $i \in \{1, \cdots, N\}$ labels the bosons) the Lagrangian [19–21]

$$\mathcal{L} = \frac{\cosh\gamma}{2} \partial_\mu\varphi^i\partial^\mu\varphi^i + \frac{\sinh\gamma}{2}\sqrt{2\left(\partial_\mu\varphi^i\partial^\nu\varphi^i\right)\left(\partial_\nu\varphi^j\partial^\mu\varphi^j\right) - \left(\partial_\mu\varphi^i\partial^\mu\varphi^i\right)^2}, \tag{4.2}$$

where we have implicit summation over $i, j = 1, \ldots, N$ which label the bosons. If $N = 1$, the theory becomes trivial and reduces to $\mathcal{L} = e^\gamma\partial_\mu\varphi\partial^\mu\varphi$. The analogue of the field strength for this theory, $\forall N \in \mathbb{N}$, is

$$\varphi_\mu{}^i \equiv \partial_\mu\varphi^i, \tag{4.3}$$

which lets us write the Lagrangian as

$$\mathcal{L} = \frac{\cosh\gamma}{2}\varphi_\mu{}^i\varphi^{\mu i} + \frac{\sinh\gamma}{2}\sqrt{2\left(\varphi_\mu{}^i\varphi^{\nu i}\right)\left(\varphi_\nu{}^j\varphi^{\mu j}\right) - \left(\varphi_\mu{}^i\varphi^{\mu i}\right)^2}, \tag{4.4}$$

where the ModMax-like structure is more apparent.

Employing the background field method [15], we split the field φ^i into a classical field C^i and a quantum field Q^i such that

$$\varphi^i = C^i + Q^i \tag{4.5}$$
$$\Rightarrow \varphi_\mu{}^i = \partial_\mu\left(C^i + Q^i\right) \tag{4.6}$$
$$\equiv C_\mu{}^i + Q_\mu{}^i, \tag{4.7}$$

where $C_\mu{}^i$ is the field strength tensor for the classical field C^i and $Q_\mu{}^i$ is the field strength tensor for the quantum field Q^i.

We notice that the analogues of the invariants S and P can thus be decomposed as

$$S \equiv \varphi_\mu{}^i\varphi^{\mu i} \tag{4.8}$$

$$= \underbrace{C_\mu{}^i C^{\mu i}}_{S_C} + 2C_\mu{}^i Q^{\mu i} + \underbrace{Q_\mu{}^i Q^{\mu i}}_{S_Q} \tag{4.9}$$

$$P \equiv \varphi_\mu{}^i \varphi^{\nu i} \tag{4.10}$$

$$= \underbrace{C_\mu{}^i C^{\nu i}}_{P_C} + Q_\mu{}^i C^{\nu i} + C_\mu{}^i Q^{\nu i} + \underbrace{Q_\mu{}^i Q^{\nu i}}_{P_Q}. \tag{4.11}$$

Note. In this chapter, we do not assume $\partial_\mu C_\mu{}^i = 0$. Nonetheless, terms linear in Q^i will vanish in the computation of the effective action, as the adjoining classical fields satisfy the equations of motion [8]. All such terms are labelled topological.

S^2 and P^2 can be written as

$$S^2 = S_C^2 + \underbrace{2S_C C_\mu{}^i Q^{\mu i}}_{\text{topological}} + 2S_C S_Q + 4C_\mu{}^i Q^{\mu i} C_\nu{}^j Q^{\nu j} + \underbrace{2S_Q C_\mu{}^i Q^{\mu i}}_{\mathcal{O}(Q^3)} + \underbrace{S_Q^2}_{\mathcal{O}(Q^4)} \tag{4.12}$$

where neglecting higher order terms we are left with

$$S^2 = S_C^2 + \underbrace{2S_C C_\mu{}^i Q^{\mu i}}_{\text{topological}} + 2S_C S_Q + 4C_\mu{}^i Q^{\mu i} C_\nu{}^j Q^{\nu j} \tag{4.13}$$

$$P^2 = C_\mu{}^i C^{\mu j} C_\nu{}^i C^{\nu j} + \underbrace{4C_\mu{}^i C^{\mu j} C^{\nu i} Q_\nu{}^j}_{\text{topological}} + 2C_\mu{}^i C^{\nu i} Q_\nu{}^j Q^{\mu j}$$
$$+ 4Q_\mu{}^i C^{\nu i} Q_\nu{}^j C^{\mu j} + 2Q_\mu{}^i C^{\nu i} C_\nu{}^j Q^{\mu j} + \mathcal{O}\left(Q^3\right) \tag{4.14}$$

Combining these terms, we see that the term underneath the square root is given by

$$\Rightarrow 2P^2 - S^2 = 2P_C^2 - S_C^2 + \underbrace{8C_\mu{}^i C^{\mu j} C^{\nu i} Q_\nu{}^j - 2S_C C_\mu{}^i Q^{\mu i}}_{\text{topological}} - 2S_C S_Q - 4C_\mu{}^i Q^{\mu i} C_\nu{}^j Q^{\nu j}$$
$$+ 4C_\mu{}^i C^{\nu i} Q_\nu{}^j Q^{\mu j} + 8Q_\mu{}^i C^{\nu i} Q_\nu{}^j C^{\mu j} + 4Q_\mu{}^i C^{\nu i} C_\nu{}^j Q^{\mu j}. \tag{4.15}$$

Taylor expanding the square root about the solely background dependent terms $2P_C^2 - S_C^2$ we see that

$$\sqrt{2P^2 - S^2} \equiv \sqrt{2P_C^2 - S_C^2 + Q} = \sqrt{2P_C^2 - S_C^2} + \frac{Q}{2\sqrt{S_C^2 + P_C^2}} - \frac{Q^2}{8\left(S_C^2 + P_C^2\right)^{\frac{3}{2}}} + \mathcal{O}\left(Q^3\right) \tag{4.16}$$

where

$$Q = \underbrace{8C_\mu{}^i C^{\mu j} C^{\nu i} Q_\nu{}^j - 2S_C C_\mu{}^i Q^{\mu i}}_{\text{topological}} - 2S_C S_Q - 4C_\mu{}^i Q^{\mu i} C_\nu{}^j Q^{\nu j}$$
$$+ 4C_\mu{}^i C^{\nu i} Q_\nu{}^j Q^{\mu j} + 8Q_\mu{}^i C^{\nu i} Q_\nu{}^j C^{\mu j} + 4Q_\mu{}^i C^{\nu i} C_\nu{}^j Q^{\mu j} \tag{4.17}$$

and up to terms quadratic in the quantum field we have

$$Q^2 = 4S_C^2 C_\mu{}^i Q^{\mu i} C_\nu{}^j Q^{\nu j} - 32 S_C C_\mu{}^i Q^{\mu i} C_\nu{}^j C^{\nu k} C^{\rho j} Q_\rho{}^k + 64\left(C_\nu{}^j C^{\nu k} C^{\rho j} Q_\rho{}^k\right)^2 + \mathcal{O}\left(Q^3\right). \tag{4.18}$$

Therefore with the classical field Lagrangian defined by

$$\mathcal{L}_C \equiv \frac{\cosh\gamma}{2} S_C + \frac{\sinh\gamma}{2}\sqrt{2P_C^2 - S_C^2} \tag{4.19}$$

we can thus write the full Lagrangian as

$$\mathcal{L} = \mathcal{L}_C + \frac{\cosh\gamma}{2} S_Q + \frac{\sinh\gamma}{2}\left(\frac{Q}{2\sqrt{2P_C^2 - S_C^2}} - \frac{Q^2}{8\left(2P_C^2 - S_C^2\right)^{\frac{3}{2}}}\right), \tag{4.20}$$

The Lagrangian for the quantum field thus suggests the quantum field has a Maxwell-like propagation $S_Q \cosh\gamma$ and an interaction vertex quadratic in both the classical and quantum fields.

We notice that we can express the quantum Lagrangian in the form

$$\mathcal{L}_Q = Q^{\mu i}\left(\frac{\cosh\gamma}{2} g_{\mu\nu}\delta^{ij} + P_{\mu\nu}{}^{ij}\right) Q^{\nu j} \tag{4.21}$$
$$= -Q^i\left(\frac{\cosh\gamma}{2}\delta^{ij}\partial^2 + \sinh\gamma\,\partial^\mu P_{\mu\nu}{}^{ij}\partial^\nu + \sinh\gamma\, P_{\mu\nu}{}^{ij}\partial^\mu\partial^\nu\right) Q^j \tag{4.22}$$

where

$$P_{\mu\nu}{}^{ij} = -\Bigg(\frac{-2S_C g_{\mu\nu}\delta^{ij} - 4C_\mu{}^i C_\nu{}^j + 4C_\mu{}^k C_\nu{}^k \delta^{ij} + 8C_\mu{}^j C_\nu{}^i + 4C_\rho{}^i C^{\rho j} g_{\mu\nu}}{4\sqrt{2P_C^2 - S_C^2}}$$
$$- \frac{4S_C^2 C_\mu{}^i C_\nu{}^j - 32 S_C C_\mu{}^i C_\rho{}^k C^{\rho j} C_\nu{}^k + 64 C_\rho{}^k C^{\rho i} C_\mu{}^k C_\tau{}^m C^{\tau j} C_\nu{}^m}{16\left(2P_C^2 - S_C^2\right)^{\frac{3}{2}}}\Bigg), \tag{4.23}$$

and we consider $P_{\mu\nu}{}^{ij}(x)$ to be a composite operator representing the cumulative effect of the classical field.

4.2 Feynman Rules

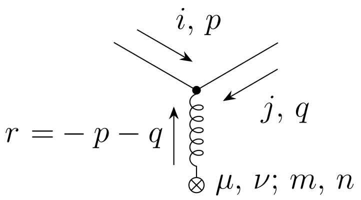
The propagator for the quantum field is

$$D^{ij} = \frac{1}{\cosh\gamma} \frac{-i\delta^{ij}}{k^2}. \tag{4.24}$$

I will draw quantum fields as solid lines and the cumulative effect of the background fields as a single coiled line.

Reading off the Lagrangian Eq. (4.22), consider first contracting Q^i with the incoming quantum field with momenta p. This means Q^j will contract with the momenta q field. The $\sim Q^i P^{ij}_{\mu\nu} \partial^\mu \partial^\nu Q^j$ term will thus carry $q^\mu q^\nu$ and the $\sim Q^i \partial^\mu P_{\mu\nu}{}^{ij} \partial^\nu Q^j$ term will contribute $r^\mu q^\nu$ as the first derivative now acts on the classical field that has momenta r.

Performing this identically for the other possible contraction, we see that the interaction vertex takes the form



$$= i \sinh\gamma \left(\delta^{im}\delta^{jn} \left(q^\mu + r^\mu\right) q^\nu + \delta^{in}\delta^{jm} \left(p^\mu + r^\mu\right) p^\nu \right) \tag{4.25}$$

$$= -i \sinh\gamma \left(\delta^{im}\delta^{jn} p^\mu q^\nu + \delta^{in}\delta^{jm} q^\mu p^\nu \right). \tag{4.26}$$

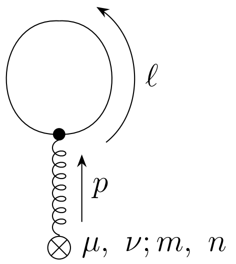
Note that $P_{\mu\nu}{}^{ij} = P_{\nu\mu}{}^{ji}$ and thus as such vertices are always contracted with these external factors we can simplify this to

$$= -2i \sinh\gamma \delta^{im}\delta^{jn} p^\mu q^\nu. \tag{4.27}$$

4.3 Background-Varying One-Loop Diagrams

As the Feynman rules are entirely analogous to the 4D ModMax case, solely with the addition of indices $i, j \in \{1, \cdots, N\}$ that sum over bosons, we can conclude immediately that if the background field is held constant, then the one loop effective action will vanish. As such, we move to the more general case, in which we do not impose, $\partial_\mu C_\nu{}^i = 0$, thus allowing the background field to arbitrarily vary.

The first diagram in the perturbative expansion is



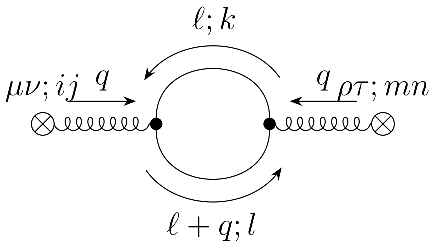
$$= \sinh\gamma \delta^{im}\delta^{jn} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \left(2i\ell^\mu \ell^\nu\right) D^{ij} \tag{4.28}$$

where the replacement $\ell^\mu \ell^\nu \to \frac{\ell^2}{4} g^{\mu\nu}$ and the propagator derived above yield

$$= -\frac{\tanh\gamma}{2}\delta^{im}\delta^{in} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} g^{\mu\nu}. \tag{4.29}$$

This integral vanishes in dimensional regularization.

The first nontrivial diagram leads to a divergent contribution with $d = 2 + 2\varepsilon$ analogous to the ModMax case with



$$= \left(\frac{1}{\varepsilon}\right) \frac{-i}{24\,(4\pi)} \Big[q^2 \left(g^{\mu\nu} g^{\rho\tau} + g^{\mu\rho} g^{\nu\tau} + g^{\mu\tau} g^{\nu\rho}\right) + 2 \left(q^\nu q^\mu g^{\tau\rho} + g^{\mu\tau} q^\nu q^\rho + g^{\nu\mu} q^\tau q^\rho + g^{\nu\rho} q^\mu q^\tau\right) + 4 \left(g^{\mu\rho} q^\nu q^\tau + g^{\nu\tau} q^\mu q^\rho\right) \Big] + \text{symmetrized indices.}$$

For the full dimensional regularization calculation of this diagram see Appendix B.

Note. The factored complete symmetrization means that all combinations of indices will appear. This leads to the latter two terms being unified as they are identical under index exchange. This leads to a coefficient of $2 + 4 = 6$.

Further, all such expressions should be bookended by

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} P_{\mu\nu}{}^{ij}\left(-q\right) P_{\rho\tau}{}^{ij}\left(q\right), \tag{4.30}$$

corresponding to the external classical fields. This prefactor which contracts the remaining free indices above is omitted above and in the calculation for brevity.

However, transforming back to real space we see that we can express the result as

$$= \left(\frac{1}{\varepsilon}\right) \frac{-i}{24\,(4\pi)} \int \mathrm{d}^2 x\, P_{\mu\nu}{}^{ij}\,(x) \left[g^{(\mu\nu} g^{\rho\tau)} \partial^2 + 6 g^{(\rho\tau} \partial^{\mu} \partial^{\nu)}\right] P_{\rho\tau}{}^{ij}\,(x)\,. \tag{4.31}$$

This bears close resemblance to the familiar $g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu$ operator with the addition of a constant and index symmetrization.

However, there is no such $\left(P_{\mu\nu}{}^{ij}\right)^2$ term in the original Lagrangian, which bodes poorly for renormalization. Namely, as $\frac{1}{\varepsilon}$ diverges in the limit $\varepsilon \to 0$, to obtain finite predictions from this theory, one would introduce a counter term to the Lagrangian, which removes this divergence. When the corrections are of the form of the original Lagrangian, this is physically well motivated as it corresponds to a redefinition of constants in the Lagrangian. However, as the form of this correction is not present in the original Lagrangian, this interpretation does not apply as the corrections are novel.

Nonetheless, I have obtained the one-loop effective action up to order $\mathcal{O}\left(\gamma^2\right)$. Truncating at an arbitrary order in γ is well motivated due to the small experimental bound of $\gamma \leq 3 \times 10^{-22}$, however for completeness we proceed with the generalization of the argument developed to an n-vertex diagram.

4.4 n Vertex Diagram

With the two-vertex diagram evaluated, to complete the one-loop effective action, we seek to evaluate all remaining diagrams containing one loop. Fortunately, there is only one diagram constructible for each number of vertices n. As such, we proceed with the generalization of the above method.

Observe that in general [8, 16], we can write the product of n propagators as

$$\prod_{i=0}^{n-1} A_i^{-1} = \int_0^1 \left(\prod_{i=0}^{n-1} \mathrm{d}x_i\right) \delta\left(\sum_{i=0}^{n-1} x_i - 1\right) \frac{(n-1)!}{\left[\sum_i x_i A_i\right]^n}. \tag{4.32}$$

For an n vertex diagram, we label the external momenta as q_i for $i \in (0, n-1)$ with momentum conservation implying $q_{n-1} = -\sum_{i=0}^{n-2} q_i$. The product of propagators inside the loop can thus be expressed as

$$\frac{(-i)^N}{\cosh^N \gamma} \prod_{i=0}^{n-1} \left(\ell + \sum_{j=1}^{i} q_j\right)^{-2}$$

$$= \frac{(-i)^N (n-1)!}{\cosh^N \gamma} \int_0^1 \left(\prod_{i=0}^{n-1} \mathrm{d}x_i \right) \delta \left(\sum_{i=0}^{n-1} x_i - 1 \right) \left[\sum_i x_i \left(\ell + \sum_{j=1}^{i} q_j \right)^2 \right]^{-n} \tag{4.33}$$

where as $\sum_i x_i = 1$, we can expand and reduce the square bracketed term to

$$\left[\sum_i x_i \left(\ell + \sum_{j=1}^{i} q_j \right)^2 \right]^{-n}$$

$$= \left[\ell^2 + \sum_i x_i \left(2\ell_\mu \sum_{j=1}^{i} q_j^\mu + \left(\sum_{j=1}^{i} q_j \right)^2 \right) \right]^{-n} \tag{4.34}$$

which we can equivalently write as

$$= \left[\left(\ell + \sum_i \sum_{j=1}^{i} x_i q_j \right)^2 - \left(\sum_i \sum_{j=1}^{i} x_i q_j \right)^2 + \sum_i x_i \left(\sum_{j=1}^{i} q_j \right)^2 \right]^{-n} \tag{4.35}$$

which under the translation $\ell \to \ell - \sum_i \sum_{j=1}^{i} x_i q_j$ becomes

$$= \left[\ell^2 - \left(\sum_i \sum_{j=1}^{i} x_i q_j \right)^2 + \sum_i x_i \left(\sum_{j=1}^{i} q_j \right)^2 \right]^{-n} \tag{4.36}$$

$$= \left[\ell^2 - \Delta^2 \right]^{-n} \tag{4.37}$$

where we identify

$$\Delta^2 = \left(\sum_i \sum_{j=1}^{i} x_i q_j \right)^2 - \sum_i x_i \left(\sum_{j=1}^{i} q_j \right)^2 . \tag{4.38}$$

The vertex factor yields factors of momenta in the numerator. As each external vertex around the loop adds q_j, and the vertex factor is the product of the momenta entering and leaving the vertex, in total we will have

$$(-i \sinh \gamma)^N \prod_{k=0}^{n-1} \left(\ell + \sum_{j=1}^{k} q_j \right)^{\alpha_{2k}} \left(\ell + \sum_{j=1}^{k} q_j \right)^{\alpha_{2k+1}} \tag{4.39}$$

where the $k = 0$ term gives us the ℓ terms (and we ignore the $\sinh^N \gamma$ prefactor for now). Under the translation identified for the denominator to be quadratic in ℓ, this numerator is translated to

$$\longrightarrow \prod_{k=0}^{n-1} \left(\ell - \sum_{i=0}^{n-1} \sum_{j=1}^{i} x_i q_j + \sum_{j=1}^{k} q_j \right)^{\alpha_{2k}} \left(\ell - \sum_{i=0}^{n-1} \sum_{j=1}^{i} x_i q_j + \sum_{j=1}^{k} q_j \right)^{\alpha_{2k+1}} . \tag{4.40}$$

We expand this product in descending powers of ℓ as only powers ℓ^{2n} and ℓ^{2n-2} will lead to divergent terms. We have that

$$= \prod_{k=0}^{n-1} \ell^{\alpha_{2k}} \ell^{\alpha_{2k+1}} + \sum_{a=0}^{2n-1} \sum_{b>a}^{2n-1} \left(\prod_{c \neq a,b}^{2n-1} \ell^{\alpha_c} \right) f\left(x,q,a\right)^{\alpha_a} f\left(x,q,b\right)^{\alpha_b} + \mathcal{O}\left(\ell^{2n-4}\right), \tag{4.41}$$

where we have defined for brevity

$$f\left(x,q,a\right)^{\alpha_a} \equiv \left(\sum_{i=0}^{n-1} \sum_{j=1}^{i} x_i q_j + \sum_{j=1}^{a} q_j \right)^{\alpha_a}. \tag{4.42}$$

Using the generalized symmetrization rule derived in Eq. (3.45)

$$\prod_{i=1}^{n} \ell^{\mu_i} \rightarrow \frac{\ell^n \left(d-2\right)!!}{2^{\frac{n}{2}} \left(d+n-2\right)!!} g^{(\mu_1\mu_2} \cdots g^{\mu_{n-1}\mu_n)} \tag{4.43}$$

where $n!! = n\left(n-2\right)\left(n-4\right)\cdots 1$ is the double factorial. This notation is cumbersome so we also write

$$\prod_{i=1}^{n} \ell^{\mu_i} \rightarrow \frac{\ell^n \left(d-2\right)!!}{2^{\frac{n}{2}} \left(d+n-2\right)!!} \bigotimes_{(\mu_i)} g. \tag{4.44}$$

This transforms the numerator to

$$\begin{aligned} &= \frac{\ell^{2n} \left(d-2\right)!!}{2^n \left(d+2n-2\right)!!} \bigotimes_{(\mu_i)} g \\ &+ \frac{\ell^{2n-2} \left(d-2\right)!!}{2^{n-1} \left(d+2n-4\right)!!} \sum_{a=0}^{2n-1} \sum_{b>a}^{2n-1} \left(\bigotimes_{(\alpha_c \neq \alpha_a, \alpha_b)} g \right) f\left(x,q,a\right)^{\alpha_a} f\left(x,q,b\right)^{\alpha_b}. \end{aligned} \tag{4.45}$$

Therefore, with the known integral

$$\int \frac{\mathrm{d}^d \ell}{\left(2\pi\right)^d} \frac{\ell^{2\beta}}{\left(\ell^2 - \Delta^2\right)^\alpha} = i\left(-1\right)^{\alpha+\beta} \frac{\Gamma\left(\beta + \frac{d}{2}\right) \Gamma\left(\alpha - \beta - \frac{d}{2}\right)}{\left(4\pi\right)^{\frac{d}{2}} \Gamma\left(\alpha\right) \Gamma\left(\frac{d}{2}\right)} \Delta^{2\left(\frac{d}{2} - \alpha + \beta\right)}, \tag{4.46}$$

we see that these two numerator terms have $\beta = n$ and $\beta = n-1$ respectively. The denominator yields $\alpha = n$. From this form it is clear to see that ℓ^{2n-4} terms and lower powers of ℓ are finite as $d \rightarrow 2$.

Proof. Such terms have $\beta = n - 2$ which would contain $\Gamma\left(\alpha - \beta - \frac{d}{2}\right)$ terms of the form

$$\Gamma\left((n) - (n-2) - \frac{d}{2}\right) = \Gamma\left(2 - \frac{d}{2}\right) \tag{4.47}$$

$$= \left(1 - \frac{d}{2}\right)\Gamma\left(1 - \frac{d}{2}\right) \tag{4.48}$$

$$= \left(1 - \frac{d}{2}\right)\left(-\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right) \tag{4.49}$$

which using $d = 2 + 2\varepsilon$ and the pole expansion of the Γ function we see that

$$= -\left(-\varepsilon\right)\left(1 + \varepsilon\right)\left(-\frac{1}{\varepsilon} - \gamma + 1\right) \tag{4.50}$$

$$= \left(1 + \varepsilon\right)\left(-1 - \varepsilon\gamma + \varepsilon\right) \tag{4.51}$$

$$= -1 + \mathcal{O}\left(\varepsilon\right), \tag{4.52}$$

which are finite and thus discarded. Lower powers of ℓ lead to larger $\alpha - \beta - \frac{d}{2}$ values which still contain the $1 - \frac{d}{2} \to -\varepsilon$ factor which cancels the poles. □

We consider only the divergent terms arising from these diagrams as they determine the observable behaviour of the theory. Finite terms can be absorbed when performing a renormalization of the theory.

As such focusing on the divergent terms, we have that the ℓ^{2n} term with $\beta = n$ becomes

$$\mathbf{C}_{2n} \equiv \frac{i\left(d-2\right)!!}{2^n\left(d + 2n - 2\right)!!}\left(\bigotimes_{(\alpha_k)} g\right)\frac{\Gamma\left(n + \frac{d}{2}\right)}{\left(4\pi\right)^{\frac{d}{2}}\Gamma\left(n\right)\Gamma\left(\frac{d}{2}\right)}\Gamma\left(-\frac{d}{2}\right)\Delta^{2\left(\frac{d}{2}\right)}, \tag{4.53}$$

and the ℓ^{2n-2} term has $\beta = n - 1$ and becomes

$$\mathbf{C}_{2n-2} \equiv \frac{i\left(d-2\right)!!}{2^{n-1}\left(d + 2n - 4\right)!!}\sum_{a=0}^{2n-1}\sum_{b>a}^{2n-1}\left(\bigotimes_{(\alpha_c \neq \alpha_a, \alpha_b)} g\right) f\left(x, q, a\right)^{\alpha_a} f\left(x, q, b\right)^{\alpha_b} \tag{4.54}$$

$$\times \frac{\Gamma\left(n - 1 + \frac{d}{2}\right)}{\left(4\pi\right)^{\frac{d}{2}}\Gamma\left(n\right)\Gamma\left(\frac{d}{2}\right)}\Gamma\left(1 - \frac{d}{2}\right)\Delta^{2\left(\frac{d}{2} - 1\right)}, \tag{4.55}$$

where we can quote the whole n vertex diagram with

$$\frac{(-2)^N}{N}\tanh^N\gamma\,(n-1)!\int_0^1\left(\prod_{i=0}^{n-1}\mathrm{d}x_i\right)\delta\left(\sum_{i=0}^{n-1}x_i-1\right)\left[\mathbf{C}_{2n}+\mathbf{C}_{2n-2}\right], \tag{4.56}$$

with the exception of the prefactor indices in the vertex factor.

In both $\mathbf{C}_{2n}$ and $\mathbf{C}_{2n-2}$ we have a polynomial in x_i of order d and a total power of q^d. In $\mathbf{C}_{2n}$, this q^d momentum dependence is contained within Δ^d, and for $\mathbf{C}_{2n-2}$, we have two uncontracted factors q^{α_a} and q^{α_b} on top of the q^{d-2}. Therefore, we conclude that in the limit of $d\to 2$, all such n vertex diagrams have the same general structure as the 2 vertex diagram with

$$\mathbf{C}_{2n}\propto q^2\left(\bigotimes_{(\alpha_k)} g\right) \tag{4.57}$$

$$\mathbf{C}_{2n-2}\propto\sum_{a=0}^{2n-1}\sum_{b>a}^{2n-1}q^{\alpha_a}q^{\alpha_b}\left(\bigotimes_{(\alpha_k\neq\alpha_a,\alpha_b)} g\right), \tag{4.58}$$

a familiar $(q^2g^{\mu\nu}-q^\mu q^\nu)$-like dependence. This structure is shrouded in the index symmetrization and x_i dependence in the full expression. As expected, while this momentum dependence is familiar, the classical field dependence that arises from the external vertices is not present in the original Lagrangian. As each n vertex diagram is divergent and has a different external field dependence (i.e. an extra external field ${P_{\mu\nu}}^{ij}(x)$), one would need to add an infinite number of terms to the Lagrangian to subtract each of these divergences. This property of requiring an infinite number of quantities to be fixed to obtain finite results is referred to as *non-renormalizability.* Such a theory is necessarily unphysical and unverifiable as it would require an infinite number of measurements to verify.

Nonetheless, ModMax and it's two dimensional analogue can still be considered on the quantum domain as *effective field theories* where one necessarily identifies a maximum energy scale of applicability. We call such a theory effective, as it is a valid low energy description, but not the true fundamental theory of the system. For example, fermions have mass in the standard model, but this is in fact an effective low energy description where the Higgs interaction which generates such mass terms has been integrated out [8].

As such, we notice that all such n vertex diagrams are proportional to $\tanh^n\gamma\sim\gamma^n$. Motivated by the small experimental bound on γ, we seek to characterize all diagrams contributing up to order $\mathcal{O}(\gamma^2)$, which is exactly all two vertex diagrams.

4.5 Two Vertex n-loop Diagrams

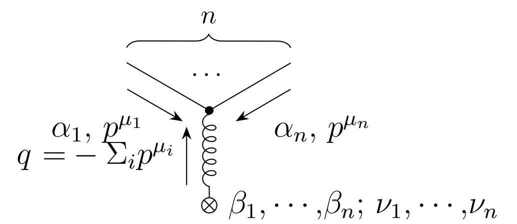
If we want to truncate at order γ^2, which corresponds to two vertices, unfortunately there is still an infinite family of diagrams that satisfy this constraint. Namely, in the expansion of the square root, we have

$$\sqrt{2P_C^2 - S_C^2 + Q} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{Q^n}{(2P_C^2 - S_C^2)^{n-\frac{1}{2}}}, \tag{4.59}$$

where Q can contain up to quartic terms in the quantum field Q_i. As such we absorb the complexities of this expansion into an object, P which captures all combinations that lead to n Q_i's in a given term with

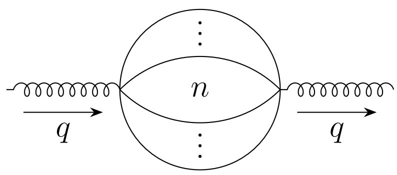
$$\sqrt{2P_C^2 - S_C^2 + Q} = \sum_{n=0}^{\infty} P^{\alpha_i \cdots \alpha_n}{}_{\mu_1 \cdots \mu_n} \prod_{i=1}^{n} \partial^{\mu_i} Q_{\alpha_i}, \tag{4.60}$$

from this expression, we read off the Feynman rules, treating $P^{\alpha_i \cdots \alpha_n}{}_{\mu_1 \cdots \mu_n}$ as a composite object representing the entire effect of the classical field (as it contains only classical fields and derivatives). As such we find that this n-vertex has Feynman rule



$$= \int \mathrm{d}^d q \, P^{\beta_i \cdots \beta_n}{}_{\nu_1 \cdots \nu_n}(q) \prod_{i=1}^{n} p^{\nu_i}. \tag{4.61}$$

Therefore, we get the two vertex n loop diagram



$$= \frac{\sinh^2\gamma}{\cosh^n\gamma} \int \mathrm{d}^d q \left(\prod_{i=1}^{n} \mathrm{d}^d \ell_i \right) \times$$
$$P^{\beta_i \cdots \beta_n}{}_{\nu_1 \cdots \nu_n}(q) \left(\prod_{i=1}^{n} \frac{p_i^{\nu_i} p_i^{\tau_i}}{p_i^2} \right) P^{\beta_i \cdots \beta_n}{}_{\tau_i \cdots \tau_n}(-q), \tag{4.62}$$

where $p_1 = q - \ell_1$, $p_i = \ell_{i-1} - \ell_i$ for $1 < i < n$ and $p_n = -\ell_{n-1}$ such that

$$\sum_{i=1}^{n} p_i = q. \tag{4.63}$$

Note. Diagrams where a loop begins and ends on the same vertex do not contribute, as they vanish in dimensional regularization. We can see this as the momentum circulating around such a loop, ℓ, will appear multiplicatively in the vertex factor in the form $\ell^\mu \ell^\nu$, and in the propagator in the form $\frac{1}{\ell^2}$. Factoring this dependence, we see the familiar

$$\int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{\ell^2} = \frac{g^{\mu\nu}}{d} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} 1, \tag{4.64}$$

which we have seen vanishes with dimensional regularization in the limit $d \to 2$, as desired.

Notice that this is an $n-1$ loop diagram hence the ℓ_i loop momenta for $1 \leq i \leq n-1$. Ignoring external momenta factors, we can write the loop integrals as

$$\propto \int \prod_{i=1}^{n-1} \mathrm{d}^d \ell_i \, \frac{p_i^{\nu_i} p_i^{\tau_i}}{p_i^2}. \tag{4.65}$$

Evaluating each loop integral in succession, beginning with ℓ_{n-1} we see (in Appendix C), that we obtain a divergence of the form

$$\propto \Gamma\left(-\frac{d}{2}\right) \ell_{n-2}^d. \tag{4.66}$$

Therefore applying this argument recursively, each successive loop integral gains an additional ℓ_i^d factor, resulting after all $n-1$ integrals, in an external momenta q dependence of

$$\propto \Gamma\left(-\frac{d}{2}\right)^{n-1} q^{d(n-1)}. \tag{4.67}$$

However, as each integral yields 6 different symmetrizations of the external indices, the exact form of an n loop diagram contains 6^n different symmetrizations and is thus challenging to write explicitly in a general form. Regardless, we can comment on the structure of the divergences present, as they are the central object of interest.

While proceeding in dimensional regularization facilitated the characterization of the divergence, it is useful to quote the dependence of such divergences on an external characteristic momenta scale, Λ. Namely, one can show that the presence of a $\frac{1}{\varepsilon}$ divergence in dimensional regularization is equivalent to a logarithmic divergence of the form

$$\frac{1}{\varepsilon} \sim \log(\Lambda). \tag{4.68}$$

Therefore, as each n loop diagram leads to a divergence of the form $\Gamma\left(-\frac{d}{2}\right)^n$, we observe logarithmic divergences of the form

$$\left(\frac{1}{\varepsilon}\right)^n q^{2n} \sim \left(q^2 \log\left(\Lambda\right)\right)^n . \tag{4.69}$$

This is a different structure of divergence for each vertex as observed for the 1 loop n vertex diagrams. However, if a pattern is hidden within these series of divergences which allows one to recover a finite number of terms by summing the series, then the theory would be more amenable to renormalization. While the background field method has proved effective in obtaining these divergences, any possible pattern is obscured by the complex index symmetrization arising from the Feynman rules in this scheme. Thus, having successfully characterized the effective action and it's divergences, we look towards an alternative quantization approach, and evaluate the possibly of hidden structures.

5
Auxiliary Fields

In our approach so far to quantizing ModMax, we have relied heavily on the background field method formalism. The background field method is powerful and effective in that we are able to consider various additional constraints on the theory such as constant classical field strength with ease. It similarly is compatible with the Taylor expansion of the square root present in ModMax and facilitates the truncation at second order in the quantum field. However, as I have demonstrated in my analysis, expanding order by order quickly becomes unfeasible.

In tackling the nonlinearity present in ModMax, the only other approach is the introduction of *auxiliary fields*. Such fields are not physical, but rather are defined in terms of the physical fields to capture some aspect of the nonlinearity. When one considers auxiliary fields, it is simple to show that such an alternative representation of the theory is equivalent to the original at the classical level. However, equivalence at the quantum level is highly nontrivial as any number of classical symmetries may be broken in either the auxiliary or original theory.

We introduce the method of auxiliary fields in the context of describing a relativistic point particle. We then outline an auxiliary field representation of ModMax, and detail the quantum behaviour of this theory. We find that this alternative approach alleviates the need for a Taylor expansion of the square root, but necessitates explicitly breaking Lorentz symmetry to proceed in the quantization. This approach is not the focus of this thesis, but is included to demonstrate the comparative effectiveness of the background field method.

5.1 Relativistic Point Particle Action

Contrary to the field theory used throughout this thesis, we now consider the action of a relativistic point particle described by a position vector $x^\mu = (t, \vec{x})$. Such an action should extremize the *proper time* between events. It can be shown that the choice

$$S = -m \int_{\text{Worldline}} \mathrm{d}l \,, \tag{5.1}$$

achieves this, where m is the mass of the particle and $\mathrm{d}l$ is the proper time between two infinitesimally separated events x^μ and $x^\mu + dx^\mu$.

Definition 6: A particle's **worldline** X^μ is a timelike path in spacetime which the particle follows.

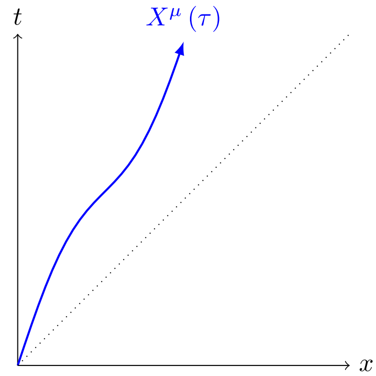


FIGURE 5.1: A worldline $X^\mu(\tau)$ parametrized by a variable τ. Notice that it is a timelike curve as it is above the null-like diagonal line.

In Minkowski space, we have that

$$\mathrm{d}l^2 = -\eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu \tag{5.2}$$

$$= \mathrm{d}t^2 - \mathrm{d}\vec{x}^2 \tag{5.3}$$

which with the parametrization $\vec{x}(t)$, we can write as

$$= \mathrm{d}t^2 - \left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)^2 \mathrm{d}t^2 \tag{5.4}$$

$$= \mathrm{d}t^2 \left(1 - \left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)^2\right) \tag{5.5}$$

$$\mathrm{d}l = \mathrm{d}t \sqrt{1 - \left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}\right)^2} \tag{5.6}$$

where we define $v^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2$ yielding

$$\mathrm{d}l = \mathrm{d}t \sqrt{1 - v^2}. \tag{5.7}$$

This allows us to write the action as

$$S = -m \int_{\text{Worldline}} \mathrm{d}t \sqrt{1 - v^2}. \tag{5.8}$$

However, this action is not clearly Lorentz invariant as we desire for a relativistic point particle. Rather than parametrizing our particle's worldline by a time t, we consider the worldline to be parameterized by a variable τ. Thus we can write the worldline of the particle as $X^\mu(\tau)$. Therefore on the worldline we can write

$$\mathrm{d}X^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \mathrm{d}\tau\,, \tag{5.9}$$

which leads to

$$\mathrm{d}l^2 = -\eta_{\mu\nu}\, \mathrm{d}X^\mu \, \mathrm{d}X^\nu \tag{5.10}$$

$$= -\eta_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} \mathrm{d}\tau^2 \tag{5.11}$$

$$\mathrm{d}l = \mathrm{d}\tau \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau}}, \tag{5.12}$$

which we can insert into the action to obtain

$$S = -m \int \mathrm{d}\tau \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau}}. \tag{5.13}$$

This is manifestly Lorentz invariant. This form of the action is nonlinear however, and defies traditional quantization techniques in the same fashion as ModMax. Therefore, we consider a representation in terms of the auxiliary field $e(\tau)$ referred to as the *Einbein action*,

$$S = \frac{1}{2} \int \mathrm{d}\tau \left(e(\tau)^{-1} \eta_{\mu\nu} \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} - e(\tau)\, m^2\right). \tag{5.14}$$

For nonzero mass m, the equation of motion for $e(\tau)$ is

$$\frac{\partial \mathcal{L}}{\partial e} = 0 \tag{5.15}$$

$$\Rightarrow e(\tau) = \frac{1}{m}\sqrt{-\eta_{\mu\nu}\frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau}}, \tag{5.16}$$

which when reinserted into the Einbein action Eq. (5.14) recovers the manifestly Lorentz invariant action in Eq. (5.13). This implies the theories are classically equivalent, but makes no predictions about their equivalence after quantization.

Note. All such actions are invariant under Lorentz transformations, translations and spatial rotations, as well as reparametrization of $\tau \to \tilde{\tau}$. While under a reparametrization of $\tau \to \tilde{\tau}$, $X^\mu(\tau)$ transforms to $\tilde{X}^\mu(\tilde{\tau})$, the auxiliary field $e(\tau)$ transforms as a density such that

$$\tilde{e}(\tilde{\tau}) = \left(\frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}\tau}\right)^{-1} e(\tau). \tag{5.17}$$

Therefore, if we view this invariance as a gauge symmetry [22], then picking a fixed value of $e(\tau)$ corresponds to fixing a gauge. Choosing $e(\tau) = 1$, the action becomes

$$S = \frac{1}{2}\int \mathrm{d}\tau\, \eta_{\mu\nu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} - m^2, \tag{5.18}$$

where the equation of motion for $e(\tau)$ can no longer be imposed, but instead becomes a constraint equation

$$1 = e(\tau) = -\frac{1}{m}\eta_{\mu\nu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} \tag{5.19}$$

$$\Rightarrow -m^2 = \eta_{\mu\nu}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}X^\nu}{\mathrm{d}\tau}, \tag{5.20}$$

which with the identification of momenta $p^\mu = \frac{\partial X^\mu}{\partial \tau}$, can be written as

$$p^2 = -m^2, \tag{5.21}$$

the familiar mass-shell condition. This form of the action is much more approachable when considering the quantization of the theory, with the only difficulty arising in imposing the constraint.

5.2 Auxiliary ModMax

We seek to apply this approach of introducing auxiliary fields, and then removing them by imposing their equation of motion as a constraint instead.

The ModMax Lagrangian has an auxiliary field representation of

$$\mathcal{L} = \cosh\gamma S + \sinh\gamma\left(S\varphi_1 + P\varphi_2\right) - \frac{1}{2}\rho^2\left(\varphi_1^2 + \varphi_2^2 - 1\right), \tag{5.22}$$

which has equations of motion for the scalar fields,

$$\sinh\gamma S = \rho^2\varphi_1 \qquad \sinh\gamma P = \rho^2\varphi_2 \qquad \rho\left(\varphi_1^2 + \varphi_2^2 - 1\right) = 0, \tag{5.23}$$

$$S = \frac{\rho^2\varphi_1}{\sinh\gamma} \qquad P = \frac{\rho^2\varphi_2}{\sinh\gamma}. \tag{5.24}$$

Notice that if $\rho = 0$, then φ_1 and φ_2 are unconstrained and the equations of motion yield $S = P = 0$, corresponding to $\mathbf{E} = \mathbf{B} = 0$. The Lagrangian then reduces to the well studied but unphysical Bialynicki-Birula theory [5].

For finite ρ, substituting the equations of motion for φ_1 and φ_2 back into the Lagrangian, we can obtain

$$\mathcal{L} = \cosh\gamma S + \frac{\sinh^2\gamma}{2}\rho^{-2}\left(S^2 + P^2\right) + \frac{1}{2}\rho^2, \tag{5.25}$$

which has equation of motion for ρ

$$\rho^4 = \sinh^2\gamma\left(S^2 + P^2\right). \tag{5.26}$$

Note. Contrary to the relativistic point particle, this action does not have reparametrization invariance. Therefore, to remove the auxiliary fields we must make use of the gauge symmetry already present. However, as S and P are gauge invariants, one does not have freedom to fix the gauge to impose the equation of motion for ρ. Nonetheless, we notice that our Lagrangian is not a function of $\partial_0 A_0$ as

$$S = -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right) \tag{5.27}$$

$$= -\frac{1}{2}\left(\partial_0 A_i \partial^0 A^i + \partial_j A_i \partial^j A^i + \partial_i A_0 \partial^i A^0 - \partial_0 A_i \partial^i A_0 - \partial_j A_i \partial^i A^j - \partial_i A_0 \partial^t A^i\right) \tag{5.28}$$

and

$$P = E^i B_i = -\left(\partial_0 A_i - \partial_i A_0\right)\varepsilon^{ijk}\partial_j A_k, \tag{5.29}$$

are functions of A_0 but not $\partial_0 A_0$. Therefore, the canonical conjugate momenta

$$\Pi^0 = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 A_0\right)} = 0, \tag{5.30}$$

vanishes, suggesting A_0 is an ill suited canonical variable. For QED, this is identically, the case, and can be treated by introducing Coulomb gauge, in which one fixes $A_0 = 0$, treating it as a non-dynamical variable.

We posit that there exists a generalized Coulomb gauge, fixing $A_0 = \omega\left(x\right)$ for some scalar function $\omega\left(x\right)$ such that we have the constraint

$$\sinh^2\gamma\left(S^2 + P^2\right) = 1, \tag{5.31}$$

namely, that $\rho = 1$ becomes fixed and non-dynamical as well. Fixing A_0 in this fashion explicitly breaks Lorentz symmetry.

Thus, imposing the resulting constraint on the Lagrangian

$$\mathcal{L} = \cosh\gamma S + \frac{\sinh^2\gamma}{2}\left(S^2 + P^2\right) + \frac{1}{2}, \tag{5.32}$$

is a classically equivalent form of ModMax. This constraint appears quite abstract, and unintuitive in comparison to the $p^2 = -m^2$ constraint that we saw for the Einbein action.

5.3 Symmetry Preservation

As ModMax has EM-duality at the level of the equations of motion, so should this equivalent formalism. Namely, in general, electromagnetic duality is expressed as

$$\begin{pmatrix} -2\frac{\partial\mathcal{L}(F')}{\partial F'_{\mu\nu}} \\ \widetilde{F}'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -2\frac{\partial\mathcal{L}(F')}{\partial F'_{\mu\nu}} \\ \widetilde{F}_{\mu\nu} \end{pmatrix}, \tag{5.33}$$

where

$$G^{\mu\nu} \equiv -2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} = \cosh\gamma F^{\mu\nu} + \rho^{-2}\sinh^2\gamma\left(SF^{\mu\nu} + P\widetilde{F}^{\mu\nu}\right). \tag{5.34}$$

An equivalent statement of electromagnetic duality [7] is if the theory satisfies

$$G_{\mu\nu}\widetilde{G}^{\mu\nu} = F_{\mu\nu}\widetilde{F}^{\mu\nu}. \tag{5.35}$$

Note. We have

$$\widetilde{G}^{\mu\nu} = \cosh\gamma \widetilde{F}^{\mu\nu} + \rho^{-2}\sinh^2\gamma \left(S\widetilde{F}^{\mu\nu} - PF^{\mu\nu}\right), \tag{5.36}$$

which reveals that

$$\begin{aligned} G_{\mu\nu}\widetilde{G}^{\mu\nu} &= \cosh^2\gamma F_{\mu\nu}\widetilde{F}^{\mu\nu} + \rho^{-2}\cosh\gamma\sinh^2\gamma\times \\ &\quad \left[\widetilde{F}_{\mu\nu}\left(SF^{\mu\nu} + P\widetilde{F}^{\mu\nu}\right) + F_{\mu\nu}\left(S\widetilde{F}^{\mu\nu} - PF^{\mu\nu}\right)\right] \\ &\quad + \rho^{-4}\sinh^4\gamma\left(SF_{\mu\nu} + P\widetilde{F}_{\mu\nu}\right)\left(S\widetilde{F}^{\mu\nu} - PF^{\mu\nu}\right) && (5.37) \\ &= -4\cosh^2\gamma P - 4\rho^{-2}\cosh\gamma\sinh^2\gamma\left[2SP - 2SP\right] \\ &\quad + 4\rho^{-4}\sinh^4\gamma\left[-S^2P + S^2P + S^2P + P^3\right] && (5.38) \\ &= -4\cosh^2\gamma P + 4\rho^{-4}\sinh^4\gamma P\left[S^2 + P^2\right] && (5.39) \end{aligned}$$

where if we impose the equation of motion for ρ,

$$\begin{aligned} &= -4\cosh^2\gamma P + 4\sinh^4\gamma P\left[\frac{1}{\sinh^2\gamma}\right] && (5.40) \\ &= 4\left(\sinh^2\gamma - \cosh^2\gamma\right)P && (5.41) \end{aligned}$$

where the hyperbolic identity $\sinh^2\gamma - \cosh^2\gamma = -1$ provides

$$= -4P \tag{5.42}$$

and as we have

$$\begin{aligned} F_{\mu\nu}\widetilde{F}^{\mu\nu} &= -4P && (5.43) \\ \Rightarrow G_{\mu\nu}\widetilde{G}^{\mu\nu} &= F_{\mu\nu}\widetilde{F}^{\mu\nu}, && (5.44) \end{aligned}$$

this theory is electromagnetically dual when the constraint is imposed as expected.

The stress energy tensor is given by

$$T_{\mu\nu} = -2\left(\frac{\partial\mathcal{L}}{\partial S}\frac{\partial S}{\partial g^{\mu\nu}} + \frac{\partial\mathcal{L}}{\partial P}\frac{\partial P}{\partial g^{\mu\nu}}\right) + g_{\mu\nu}\mathcal{L}, \tag{5.45}$$

where

$$\frac{\partial\mathcal{L}}{\partial S} = \cosh\gamma + \rho^{-2}\sinh^2\gamma S \qquad \frac{\partial\mathcal{L}}{\partial P} = \rho^{-2}\sinh^2\gamma P \tag{5.46}$$

and

$$\frac{\partial S}{\partial g_{\mu\nu}} = -\frac{1}{2}F_\mu{}^\rho F_{\nu\rho} \qquad \frac{\partial P}{\partial g_{\mu\nu}} = -\frac{1}{4}\left(F_\mu{}^\rho\widetilde{F}_{\mu\rho} + F_\nu{}^\rho\widetilde{F}_{\mu\rho}\right), \tag{5.47}$$

which lead to a trace of the form

$$
\begin{aligned}
T^{\mu}{}_{\mu} &= -4\left(S\frac{\partial\mathcal{L}}{\partial S} + P\frac{\partial\mathcal{L}}{\partial P} - \mathcal{L}\right) &(5.48)\\
&= -4\Bigg(S\cosh\gamma + \rho^{-2}S^2\sinh^2\gamma + \rho^{-2}P^2\sinh^2\gamma - S\cosh\gamma\\
&\qquad - \rho^{-2}\frac{\sinh^2\gamma}{2}\left(S^2+P^2\right) - \frac{1}{2}\rho^2\Bigg) &(5.49)\\
&= -4\left(\rho^{-2}\frac{\sinh^2\gamma}{2}\left(S^2+P^2\right) - \frac{1}{2}\rho^2\right) &(5.50)
\end{aligned}
$$

where we see the equation of motion Eq. (5.26) causes this term to exactly vanish

$$
T^{\mu}{}_{\mu} = 0, \tag{5.51}
$$

as desired.

Therefore, we have that this auxiliary field representation of ModMax maintains both the symmetries of the original theory at the classical level. Note that this makes no comment on the equivalence at the quantum level, which is significantly inhibited by the breaking of Lorentz symmetry by fixing A_0. This digression suggests that one cannot integrate out ρ without explicitly breaking Lorentz symmetry (or purely recovering ModMax itself). This contrasts with the effectiveness of the background field method, in which we were able to preserve Lorentz invariance in our quantization procedure. Alternative auxiliary field representations are a natural extension of this work, however are likely less elucidating of underlying structures than the background field method developed above.

6

Conclusion

The central aim of this project was to quantize ModMax by obtaining the effective action. While ModMax's nonlinearity poses a great resistance to traditional quantization techniques, the background field method and dimensional regularization proved highly effective in quantizing this theory. As such, I achieved the central aim of this project by characterizing the effective action arising from quantum corrections in both a static and varying classical background field. I obtained the effective action by evaluating all one loop Feynman diagrams and all two loop diagrams containing up to two vertices.

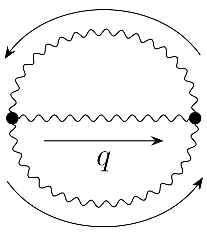
This effective action provides corrections to the classical theory arising from the quantum domain. I showed that these corrections exactly vanish when the background field is static. This suggests that under this restriction, there are no quantum corrections to this theory, a novel result undiscovered in literature. However, when the background field was allowed to vary, divergent corrections arose which were not of the form of the original Lagrangian. This was also a novel result. While these corrections appear to respect conformal symmetry, as ModMax is the unique nonlinear theory possessing conformal symmetry and electromagnetic duality, this suggests that these corrections must break electromagnetic duality. While this result hints towards the non-physicality of ModMax, it is still a valid effective field theory. Further, the discovery of novel conformal field theories is of great theoretical interest. The natural extension of this investigation would be to investigate the properties of the classical conformal theory generated by these quantum corrections.

This result motivated the study of the two dimensional analogue of ModMax, due to the increased predictive power of conformal symmetry in two dimensions. I applied the method I developed to quantize ModMax to it's two dimensional analogue theory. As expected, I obtained corrections of an analogous form, also vanishing when the background field is held constant. Allowing the background to vary, I similarly obtained divergent quantum corrections to the classical theory that are not of the form of the initial Lagrangian.

For both of the theories investigated, while the quantum corrections obtained were novel results, the divergence and new form of these corrections suggest that the theories do not admit physical quantum versions in their current form. Nonetheless, they are valid as effective field theories, and the possibility of generating new conformal field theories through the corrections obtained is promising. In the landscape of nonlinear electrodynamics, this quantization of ModMax serves to demonstrate the possibility of translating such classical nonlinear theories to the quantum domain.

A
Nontrivial 2-Loop Diagram

We proceed in dimensional regularization with $d \neq 4$ in which the diagram thus evaluates to



$$= \frac{\sinh^2\gamma}{6} B^{\mu_1}{}_{\nu_1}{}^{\rho_1}{}_{\tau_1}{}^{\alpha_1}{}_{\beta_1} B^{\mu_2}{}_{\nu_2}{}^{\rho_2}{}_{\tau_2}{}^{\alpha_2}{}_{\beta_2} \times$$

$$\int \frac{\mathrm{d}^d p\, \mathrm{d}^d q}{(2\pi)^{2d}} p_{\mu_1} q_{\rho_1} (p-q)_{\alpha_1} p_{\mu_2} q_{\rho_2} (p-q)_{\alpha_2} D^{\nu_1\nu_2} D^{\tau_1\tau_2} D^{\beta_1\beta_2} \tag{A.1}$$

$$= (-i)^3 \frac{\sinh^2\gamma}{6\cosh^3\gamma} B^{\mu_1}{}_{\nu_1}{}^{\rho_1}{}_{\tau_1}{}^{\alpha_1}{}_{\beta_1} B^{\mu_2\nu_1\rho_2\tau_1\alpha_2\beta_1} \times \tag{A.2}$$

$$\int \frac{\mathrm{d}^d p\, \mathrm{d}^d q}{(2\pi)^{2d}} \frac{p_{\mu_1} q_{\rho_1} (p-q)_{\alpha_1} p_{\mu_2} q_{\rho_2} (p-q)_{\alpha_2}}{p^2 q^2 (p-q)^2}. \tag{A.3}$$

The numerator has terms with either $2, 3$ or 4 factors of p in it (and symmetrically for q). Inspecting the $p_{\mu_1}p_{\mu_2}$ term of the integral, we have

$$\int \frac{\mathrm{d}^d p\, \mathrm{d}^d q}{(2\pi)^{2d}} \frac{p_{\mu_1}p_{\mu_2}q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{p^2 q^2 \left(p^2 - 2p^\mu q_\mu + q^2\right)} \tag{A.4}$$

where focusing on the p integral, we write

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{p_{\mu_1}p_{\mu_2}}{p^2 \left(p^2 - 2p^\mu q_\mu + q^2\right)} \tag{A.5}$$

and introduce a Feynman integral over x

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{p_{\mu_1}p_{\mu_2}}{\left[p^2 x + \left(p^2 - 2p^\mu q_\mu + q^2\right)(1-x)\right]^2} \tag{A.6}$$

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{p_{\mu_1}p_{\mu_2}}{\left[p^2 + \left(-2p^\mu q_\mu + q^2\right)(1-x)\right]^2} \tag{A.7}$$

where we complete the square in the denominator

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{p_{\mu_1}p_{\mu_2}}{\left[\left(p_\mu - q_\mu(1-x)\right)^2 - q^2(1-x)^2\right]^2} \tag{A.8}$$

and make the translation $p_\mu \to p_\mu + q_\mu(1-x)$,

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{\left(p_{\mu_1} + q_{\mu_1}(1-x)\right)\left(p_{\mu_2} + q_{\mu_2}(1-x)\right)}{\left[p^2 - q^2(1-x)^2\right]^2} \tag{A.9}$$

where here the cross terms in the integrand with one p will vanish, and thus we are left with

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{p_{\mu_1}p_{\mu_2} + q_{\mu_1}q_{\mu_2}(1-x)^2}{\left[p^2 - q^2(1-x)^2\right]^2} \tag{A.10}$$

where we make use of the known integral (A.4 in [16])

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{p^{2\beta}}{\left(p^2 - \Delta^2\right)^\alpha} = i(-1)^{\alpha+\beta} \frac{\Gamma\left(\beta + \frac{d}{2}\right)\Gamma\left(\alpha - \beta - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma(\alpha)\Gamma\left(\frac{d}{2}\right)} \Delta^{2\left(\frac{d}{2} - \alpha + \beta\right)} \tag{A.11}$$

with $\alpha = 2$, $\beta = 0, 1$ from $p_{\mu_1}p_{\mu_2} \to \frac{p^2}{4} g_{\mu_1\mu_2}$ and $\Delta^2 = q^2(1-x)^2$ to obtain

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2} \int_0^1 \mathrm{d}x\, i\left[q_{\mu_1}q_{\mu_2}(1-x)^2 \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \Delta^{d-4}\right.$$

$$-\frac{g_{\mu_1\mu_2}}{4}\frac{\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\Delta^{d-2}\Bigg] \tag{A.12}$$

where substituting in Δ^2 and using $\Gamma(x+1) = x\Gamma(x) \Rightarrow \Gamma\left(1+\frac{d}{2}\right) = \frac{d}{2}\Gamma\left(\frac{d}{2}\right)$ leads to

$$\begin{aligned}= \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2}\int_0^1 \mathrm{d}x\,\frac{i}{(4\pi)^{\frac{d}{2}}}\Bigg[&q_{\mu_1}q_{\mu_2}q^{d-4}(1-x)^{2+d-4}\,\Gamma\left(2-\frac{d}{2}\right)\\ &-\frac{d}{2}\frac{g_{\mu_1\mu_2}}{4}\Gamma\left(1-\frac{d}{2}\right)q^{d-2}(1-x)^{d-2}\Bigg]\end{aligned} \tag{A.13}$$

where evaluating the Feynman integral gives $\frac{1}{d-1}$ and thus

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2}\frac{i}{(4\pi)^{\frac{d}{2}}(d-1)}\left[q_{\mu_1}q_{\mu_2}q^{d-4}\Gamma\left(2-\frac{d}{2}\right)-\frac{dg_{\mu_1\mu_2}}{8}\Gamma\left(1-\frac{d}{2}\right)q^{d-2}\right] \tag{A.14}$$

where lastly with $\Gamma\left(2-\frac{d}{2}\right) = \left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)$ we arrive at

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2}\frac{i\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}(d-1)}\left[q_{\mu_1}q_{\mu_2}q^{d-4}\left(1-\frac{d}{2}\right)-\frac{dg_{\mu_1\mu_2}}{8}q^{d-2}\right], \tag{A.15}$$

where $\forall d \neq 4$, we are left with a symmetrizable integral over q that will vanish identically. By analytic continuation, in this regularization scheme we conclude that the integrals also vanish at $d = 4$.

The other two possible numerators with 3 and 4 factors of p follow similarly as the Feynman integral substitution in the denominator is independent of the momenta in the numerator. Namely, for 3 factors of p, we have

$$\int \frac{\mathrm{d}^d p\,\mathrm{d}^d q}{(2\pi)^{2d}}\frac{p_{\mu_1}p_{\mu_2}p_{\alpha_1}q_{\rho_1}q_{\rho_2}q_{\alpha_2}}{p^2q^2(p^2-2p^\mu q_\mu)} + \alpha_1 \leftrightarrow \alpha_2. \tag{A.16}$$

Focusing on the p integral, the same process and translation $p_\mu \to p_\mu + xq_\mu$ yields

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{q_{\rho_1}q_{\rho_2}q_{\alpha_2}}{q^2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{(p_{\mu_1}+xq_{\mu_1})(p_{\mu_2}+xq_{\mu_2})(p_{\alpha_1}+xq_{\alpha_1})}{[p^2-q^2x^2]^2} \tag{A.17}$$

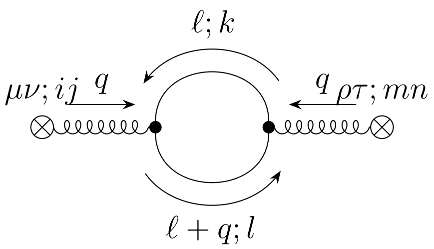
$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{q_{\rho_1}q_{\rho_2}q_{\alpha_1}q_{\alpha_2}}{q^2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{p_{\mu_1}p_{\mu_2}+x^2q_{\mu_1}q_{\mu_2}}{[p^2-q^2x^2]^2}, \tag{A.18}$$

which is in fact exactly as we saw for the 4 factors of p case. Likewise by symmetry, the 2 factors will be identical to the 4 factors under exchange of $p \leftrightarrow q$. Thus it is sufficient to multiply our result by 3 to account for all terms.

B

Scalar Field Nontrivial Diagram

The first nontrivial diagram is



$$
\begin{aligned}
&= \frac{\tanh^2 \gamma}{2} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{\ell^2 \, (\ell + q)^2} \times \\
&\quad \left(\delta^{ik} \delta^{jl} \ell^\mu \, (\ell + q)^\nu + \delta^{il} \delta^{jk} \, (\ell + q)^\mu \, \ell^\nu\right) \\
&\quad \left(\delta^{km} \delta^{ln} \ell^\rho \, (\ell + q)^\tau + \delta^{kn} \delta^{lm} \, (\ell + q)^\rho \, \ell^\tau\right) \qquad \text{(B.1)} \\
&= \frac{\tanh^2 \gamma}{2} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{\ell^2 \, (\ell + q)^2} \times \\
&\quad \Bigg(\delta^{im} \delta^{jn} \left((\ell + q)^\nu \ell^\mu (\ell + q)^\tau \ell^\rho + (\ell + q)^\mu \, \ell^\nu \, (\ell + q)^\rho \, \ell^\tau \right) \\
&\qquad + \delta^{in} \delta^{jm} \left((\ell + q)^\nu \ell^\mu (\ell + q)^\rho \ell^\tau + (\ell + q)^\mu \, \ell^\nu \ell^\rho \, (\ell + q)^\tau \right) \Bigg) \qquad \text{(B.2)}
\end{aligned}
$$

Denoting index exchange by $\{\rho \leftrightarrow \tau\}$, allows us to write this as

$$= \tanh^2 \gamma \left(\delta^{im} \delta^{jn} + \delta^{in} \delta^{jm} \{\rho \leftrightarrow \tau\} \right) (1 + \{\mu, \rho \leftrightarrow \nu, \tau\}) \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{(\ell + q)^\nu \ell^\mu (\ell + q)^\tau \ell^\rho}{\ell^2 (\ell + q)^2} \tag{B.3}$$

Ignoring the prefactor and index exchange for now, we are left with an integral that expands to

$$\int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{\ell^\nu \ell^\mu \ell^\tau \ell^\rho + \ell^\nu \ell^\mu q^\tau \ell^\rho + q^\nu \ell^\mu \ell^\tau \ell^\rho + q^\nu \ell^\mu q^\tau \ell^\rho}{\ell^2 (\ell + q)^2} \tag{B.4}$$

where introducing a Feynman integral for the denominator we see that

$$= \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{\ell^\nu \ell^\mu \ell^\tau \ell^\rho + \ell^\nu \ell^\mu q^\tau \ell^\rho + q^\nu \ell^\mu \ell^\tau \ell^\rho + q^\nu \ell^\mu q^\tau \ell^\rho}{\left[\ell^2 (1 - x) + x (\ell + q)^2 \right]^2} \tag{B.5}$$

$$= \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{\ell^\nu \ell^\mu \ell^\tau \ell^\rho + \ell^\nu \ell^\mu q^\tau \ell^\rho + q^\nu \ell^\mu \ell^\tau \ell^\rho + q^\nu \ell^\mu q^\tau \ell^\rho}{\left[\ell^2 + x (2\ell_\mu q^\mu + q^2) \right]^2} \tag{B.6}$$

Completing the square by adding and subtracting $q^2 x^2$ we see that we can factor

$$= \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{\ell^\nu \ell^\mu \ell^\tau \ell^\rho + \ell^\nu \ell^\mu q^\tau \ell^\rho + q^\nu \ell^\mu \ell^\tau \ell^\rho + q^\nu \ell^\mu q^\tau \ell^\rho}{\left[(\ell^\mu + x q^\mu)^2 + x (1 - x) q^2 \right]^2} \tag{B.7}$$

and translating $\ell^\mu \to \ell^\mu - x q^\mu$, as the denominator becomes even in ℓ, odd numerator terms vanish leaving

$$\begin{aligned} = \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \int_0^1 \mathrm{d}x \, & \frac{\ell^\nu \ell^\mu \ell^\tau \ell^\rho}{\left[\ell^2 + q^2 x (1 - x) \right]^2} + \frac{[x^2] \, \ell^\nu q^\mu \ell^\tau q^\rho}{\left[\ell^2 + q^2 x (1 - x) \right]^2} \\ & + \frac{[x^2 - x] \left(q^\nu q^\mu \ell^\tau \ell^\rho + q^\nu \ell^\mu \ell^\tau q^\rho + \ell^\nu \ell^\mu q^\tau q^\rho + \ell^\nu q^\mu q^\tau \ell^\rho \right)}{\left[\ell^2 + q^2 x (1 - x) \right]^2} \\ & + \frac{[x^2 - 2x + 1] \, q^\nu \ell^\mu q^\tau \ell^\rho}{\left[\ell^2 + q^2 x (1 - x) \right]^2} + \frac{[x^4 - 2x^3 + x^2] \, q^\nu q^\mu q^\tau q^\rho}{\left[\ell^2 + q^2 x (1 - x) \right]^2} \end{aligned} \tag{B.8}$$

where we can factor the polynomials into

$$= \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{\ell^\nu \ell^\mu \ell^\tau \ell^\rho}{\left[\ell^2 + q^2 x (1 - x) \right]^2} + \frac{[x^2] \, \ell^\nu q^\mu \ell^\tau q^\rho}{\left[\ell^2 + q^2 x (1 - x) \right]^2}$$

$$+\frac{[x(1-x)]\left(q^\nu q^\mu \ell^\tau \ell^\rho + q^\nu \ell^\mu \ell^\tau q^\rho + \ell^\nu \ell^\mu q^\tau q^\rho + \ell^\nu q^\mu q^\tau \ell^\rho\right)}{\left[\ell^2 + q^2 x(1-x)\right]^2}$$
$$+\frac{\left[(1-x)^2\right] q^\nu \ell^\mu q^\tau \ell^\rho}{\left[\ell^2 + q^2 x(1-x)\right]^2} + \frac{\left[x^2(1-x)^2\right] q^\nu q^\mu q^\tau q^\rho}{\left[\ell^2 + q^2 x(1-x)\right]^2} \tag{B.9}$$

by symmetry, we make use of

$$\ell^\mu \ell^\nu \to \frac{1}{d}\ell^2 g^{\mu\nu} \tag{B.10}$$
$$\ell^\mu \ell^\nu \ell^\rho \ell^\tau \to \frac{1}{d(d+2)}\ell^4\left(g^{\mu\nu}g^{\rho\tau} + g^{\mu\rho}g^{\nu\tau} + g^{\mu\tau}g^{\nu\rho}\right) \tag{B.11}$$

where this reduces to

$$= \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\int_0^1 \mathrm{d}x\, \frac{\ell^4}{d(d+2)}\frac{g^{\mu\nu}g^{\rho\tau} + g^{\mu\tau}g^{\nu\rho} + g^{\mu\rho}g^{\nu\tau}}{\left[\ell^2 + q^2 x(1-x)\right]^2}$$
$$+\frac{\ell^2}{d}\frac{\left[x^2\right] g^{\nu\tau}q^\mu q^\rho + [x(1-x)]\left(q^\nu q^\mu g^{\tau\rho} + g^{\mu\tau}q^\nu q^\rho + g^{\nu\mu}q^\tau q^\rho + g^{\nu\rho}q^\mu q^\tau\right)}{\left[\ell^2 + q^2 x(1-x)\right]^2}$$
$$+\frac{\ell^2}{d}\frac{\left[(1-x)^2\right] g^{\mu\rho}q^\nu q^\tau}{\left[\ell^2 + q^2 x(1-x)\right]^2} + \frac{\left[x^2(1-x)^2\right] q^\nu q^\mu q^\tau q^\rho}{\left[\ell^2 + q^2 x(1-x)\right]^2} \tag{B.12}$$

where we make use of the known integral (A.4 in [16]) with $\Delta^2 = -q^2 x(1-x)$,

$$\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{\ell^{2\beta}}{\left(\ell^2 - \Delta^2\right)^\alpha} = i(-1)^{\alpha+\beta}\frac{\Gamma\left(\beta + \frac{d}{2}\right)\Gamma\left(\alpha - \beta - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma(\alpha)\Gamma\left(\frac{d}{2}\right)}\Delta^{2\left(\frac{d}{2}-\alpha+\beta\right)} \tag{B.13}$$

$$= \frac{i}{(4\pi)^{\frac{d}{2}}\Gamma(2)\Gamma\left(\frac{d}{2}\right)}\int_0^1 \mathrm{d}x$$
$$\frac{\Delta^d}{d(d+2)}\Gamma\left(2+\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)\left(g^{\mu\nu}g^{\rho\tau} + g^{\mu\rho}g^{\nu\tau} + g^{\mu\tau}g^{\nu\rho}\right)$$
$$+\frac{\Delta^{d-2}}{d}\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)[x(1-x)]\times$$
$$\left(q^\nu q^\mu g^{\tau\rho} + g^{\mu\tau}q^\nu q^\rho + g^{\nu\mu}q^\tau q^\rho + g^{\nu\rho}q^\mu q^\tau\right)$$
$$+\frac{\Delta^{d-2}}{d}\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\left[x^2\right] g^{\nu\tau}q^\mu q^\rho$$
$$+\frac{\Delta^{d-2}}{d}\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\left[(1-x)^2\right] g^{\mu\rho}q^\nu q^\tau$$
$$+\Delta^{d-4}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\left[x^2(1-x)^2\right] q^\nu q^\mu q^\tau q^\rho \tag{B.14}$$

where we can simplify using $\Gamma(1+x) = x\Gamma(x)$

$$
\begin{aligned}
&= \frac{i\Gamma\left(-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 \mathrm{d}x \\
&\frac{\Delta^d}{d\,(d+2)}\left(1+\frac{d}{2}\right)\left(\frac{d}{2}\right)\left(g^{\mu\nu}g^{\rho\tau}+g^{\mu\rho}g^{\nu\tau}+g^{\mu\tau}g^{\nu\rho}\right) \\
&+\frac{\Delta^{d-2}}{d}\left(\frac{d}{2}\right)\left(-\frac{d}{2}\right)\left[x\,(1-x)\right]\times \\
&\left(q^\nu q^\mu g^{\tau\rho}+g^{\mu\tau}q^\nu q^\rho+g^{\nu\mu}q^\tau q^\rho+g^{\nu\rho}q^\mu q^\tau\right) \\
&+\frac{\Delta^{d-2}}{d}\left(\frac{d}{2}\right)\left(-\frac{d}{2}\right)\left[x^2\right]g^{\nu\tau}q^\mu q^\rho \\
&+\frac{\Delta^{d-2}}{d}\left(\frac{d}{2}\right)\left(-\frac{d}{2}\right)\left[(1-x)^2\right]g^{\mu\rho}q^\nu q^\tau \\
&+\Delta^{d-4}\left(1-\frac{d}{2}\right)\left(-\frac{d}{2}\right)\left[x^2\,(1-x)^2\right]q^\nu q^\mu q^\tau q^\rho
\end{aligned}
\tag{B.15}
$$

which further reduces to

$$
\begin{aligned}
&= \frac{i\Gamma\left(-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 \mathrm{d}x \\
&\frac{\Delta^d}{4}\left(g^{\mu\nu}g^{\rho\tau}+g^{\mu\rho}g^{\nu\tau}+g^{\mu\tau}g^{\nu\rho}\right) \\
&-\frac{d\Delta^{d-2}}{4}\left[x\,(1-x)\right]\left(q^\nu q^\mu g^{\tau\rho}+g^{\mu\tau}q^\nu q^\rho+g^{\nu\mu}q^\tau q^\rho+g^{\nu\rho}q^\mu q^\tau\right) \\
&-\frac{d\Delta^{d-2}}{4}\left[x^2\right]g^{\nu\tau}q^\mu q^\rho-\frac{d\Delta^{d-2}}{4}\left[(1-x)^2\right]g^{\mu\rho}q^\nu q^\tau \\
&+\frac{d\,(d-2)\,\Delta^{d-4}}{4}\left[x^2\,(1-x)^2\right]q^\nu q^\mu q^\tau q^\rho
\end{aligned}
\tag{B.16}
$$

where substituting in $\Delta^2 = -q^2 x\,(1-x)$, we see all polynomials are of order d as expected with

$$
\begin{aligned}
&= \frac{i\Gamma\left(-\frac{d}{2}\right)}{4\,(4\pi)^{\frac{d}{2}}} \int_0^1 \mathrm{d}x \\
&q^d\left[-x\,(1-x)\right]^{\frac{d}{2}}\left(g^{\mu\nu}g^{\rho\tau}+g^{\mu\rho}g^{\nu\tau}+g^{\mu\tau}g^{\nu\rho}\right) \\
&+dq^{d-2}\left[-x\,(1-x)\right]^{\frac{d}{2}}\left(q^\nu q^\mu g^{\tau\rho}+g^{\mu\tau}q^\nu q^\rho+g^{\nu\mu}q^\tau q^\rho+g^{\nu\rho}q^\mu q^\tau\right) \\
&-dq^{d-2}\left[(-x)^{\frac{d}{2}+1}(1-x)^{\frac{d}{2}-1}\right]g^{\nu\tau}q^\mu q^\rho
\end{aligned}
$$

$$- dq^{d-2}\left[(-x)^{\frac{d}{2}-1}(1-x)^{\frac{d}{2}+1}\right] g^{\mu\rho}q^{\nu}q^{\tau}$$
$$+ d(d-2)\, q^{d-4}\left[-x(1-x)\right]^{\frac{d}{2}} q^{\nu}q^{\mu}q^{\tau}q^{\rho} \tag{B.17}$$

and evaluating the Feynman integrals using (A.3 in [16])

$$\int_0^1 \mathrm{d}x\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

we arrive at

$$\begin{aligned}
= \frac{i(-1)^{\frac{d}{2}}\Gamma\left(-\frac{d}{2}\right)}{4(4\pi)^{\frac{d}{2}}}\Bigg[\\
q^{d}\frac{\Gamma\left(\frac{d}{2}+1\right)^2}{\Gamma(d+2)}\left(g^{\mu\nu}g^{\rho\tau}+g^{\mu\rho}g^{\nu\tau}+g^{\mu\tau}g^{\nu\rho}\right) \\
+ dq^{d-2}\frac{\Gamma\left(\frac{d}{2}+1\right)^2}{\Gamma(d+2)}\left(q^{\nu}q^{\mu}g^{\tau\rho}+g^{\mu\tau}q^{\nu}q^{\rho}+g^{\nu\mu}q^{\tau}q^{\rho}+g^{\nu\rho}q^{\mu}q^{\tau}\right) \\
+ dq^{d-2}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+2\right)}{\Gamma(d+2)}\left[g^{\mu\rho}q^{\nu}q^{\tau}+g^{\nu\tau}q^{\mu}q^{\rho}\right] \\
+ d(d-2)\, q^{d-4}\frac{\Gamma\left(\frac{d}{2}+1\right)^2}{\Gamma(d+2)}q^{\nu}q^{\mu}q^{\tau}q^{\rho}\Bigg]
\end{aligned} \tag{B.18}$$

where factoring out the Gamma functions allows us to conclude with

$$\begin{aligned}
= \frac{i(-1)^{\frac{d}{2}}\Gamma\left(-\frac{d}{2}\right)}{4(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(\frac{d}{2}+1\right)^2}{\Gamma(d+2)}\Bigg[\\
q^{d}\left(g^{\mu\nu}g^{\rho\tau}+g^{\mu\rho}g^{\nu\tau}+g^{\mu\tau}g^{\nu\rho}\right) \\
+ dq^{d-2}\left(q^{\nu}q^{\mu}g^{\tau\rho}+g^{\mu\tau}q^{\nu}q^{\rho}+g^{\nu\mu}q^{\tau}q^{\rho}+g^{\nu\rho}q^{\mu}q^{\tau}\right) \\
+ (d+2)\, q^{d-2}\left(g^{\mu\rho}q^{\nu}q^{\tau}+g^{\nu\tau}q^{\mu}q^{\rho}\right) \\
+ d(d-2)\, q^{d-4}q^{\nu}q^{\mu}q^{\tau}q^{\rho}\Bigg]
\end{aligned} \tag{B.19}$$

Notice that in the limit of $d = 2 + 2\varepsilon$ with $\varepsilon \to 0$ we have

$$\begin{aligned}
= \frac{i(-1)^{1+\varepsilon}\Gamma(-1-\varepsilon)}{4(4\pi)^{1+\varepsilon}}\frac{\Gamma(1+\varepsilon)^2}{\Gamma(4+\varepsilon)}\Bigg[\\
q^2q^{2\varepsilon}\left(g^{\mu\nu}g^{\rho\tau}+g^{\mu\rho}g^{\nu\tau}+g^{\mu\tau}g^{\nu\rho}\right) \\
+ 2(1+\varepsilon)\, q^{2\varepsilon}\left(q^{\nu}q^{\mu}g^{\tau\rho}+g^{\mu\tau}q^{\nu}q^{\rho}+g^{\nu\mu}q^{\tau}q^{\rho}+g^{\nu\rho}q^{\mu}q^{\tau}\right)
\end{aligned}$$

$$+ (4 + 2\varepsilon) q^{2\varepsilon} (g^{\mu\rho} q^{\nu} q^{\tau} + g^{\nu\tau} q^{\mu} q^{\rho})$$
$$+ 4\varepsilon (1 + \varepsilon) q^{-2(1+\varepsilon)} q^{\nu} q^{\mu} q^{\tau} q^{\rho} \Big] \tag{B.20}$$

and with $\Gamma(-1-\varepsilon) = \frac{1}{\varepsilon} - \gamma + 1 + \mathcal{O}(\varepsilon)$, considering only divergent terms, we can discard the last term

$$= \left(\frac{1}{\varepsilon}\right) \frac{-i}{24 (4\pi)} \Big[$$
$$q^2 (g^{\mu\nu} g^{\rho\tau} + g^{\mu\rho} g^{\nu\tau} + g^{\mu\tau} g^{\nu\rho})$$
$$+ 2 (q^{\nu} q^{\mu} g^{\tau\rho} + g^{\mu\tau} q^{\nu} q^{\rho} + g^{\nu\mu} q^{\tau} q^{\rho} + g^{\nu\rho} q^{\mu} q^{\tau})$$
$$+ 4 (g^{\mu\rho} q^{\nu} q^{\tau} + g^{\nu\tau} q^{\mu} q^{\rho}) \Big].$$

C

Scalar Field N-Loop 2-Vertex Calculation

Focusing on ℓ_{n-1}, we see that it's dependence can be factored and evaluated with

$$\propto \int \mathrm{d}^4\ell_{n-1}\, \frac{\ell_{n-1}^{\nu_n}\ell_{n-1}^{\tau_n}\left(\ell_{n-1}-\ell_{n-2}\right)^{\nu_{n-1}}\left(\ell_{n-1}-\ell_{n-2}\right)^{\tau_{n-1}}}{\ell_{n-1}^2\left(\ell_{n-1}-\ell_{n-2}\right)^2} \tag{C.1}$$

we notice this is exactly of the form of the 1-loop 2-vertex diagram already evaluated. Proceeding identically, we begin with a Feynman integral in the denominator

$$= \int \mathrm{d}^4\ell_{n-1}\,\mathrm{d}x\, \frac{\ell_{n-1}^{\nu_n}\ell_{n-1}^{\tau_n}\left(\ell_{n-1}-\ell_{n-2}\right)^{\nu_{n-1}}\left(\ell_{n-1}-\ell_{n-2}\right)^{\tau_{n-1}}}{\left[(1-x)\,\ell_{n-1}^2+x\left(\ell_{n-1}-\ell_{n-2}\right)^2\right]^2} \tag{C.2}$$

$$= \int \mathrm{d}^4\ell_{n-1}\,\mathrm{d}x\, \frac{\ell_{n-1}^{\nu_n}\ell_{n-1}^{\tau_n}\left(\ell_{n-1}-\ell_{n-2}\right)^{\nu_{n-1}}\left(\ell_{n-1}-\ell_{n-2}\right)^{\tau_{n-1}}}{\left[\ell_{n-1}^2+x\left(2\ell_{n-1}\cdot\ell_{n-2}-\ell_{n-2}^2\right)\right]^2} \tag{C.3}$$

$$= \int \mathrm{d}^4\ell_{n-1}\,\mathrm{d}x\, \frac{\ell_{n-1}^{\nu_n}\ell_{n-1}^{\tau_n}\left(\ell_{n-1}-\ell_{n-2}\right)^{\nu_{n-1}}\left(\ell_{n-1}-\ell_{n-2}\right)^{\tau_{n-1}}}{\left[\left(\ell_{n-1}+x\ell_{n-2}\right)^2-x^2\ell_{n-2}^2+x\ell_{n-2}^2\right]^2} \tag{C.4}$$

where translating $\ell_{n-1}\to\ell_{n-1}-x\ell_{n-2}$

$$= \int \mathrm{d}^4\ell_{n-1}\,\mathrm{d}x\, \frac{\left(\ell_{n-1}-x\ell_{n-2}\right)^{\nu_n}\left(\ell_{n-1}-x\ell_{n-2}\right)^{\tau_n}\left(\ell_{n-1}-(1-x)\,\ell_{n-2}\right)^{\nu_{n-1}}\left(\ell_{n-1}-(1-x)\,\ell_{n-2}\right)^{\tau_{n-1}}}{\left[\ell_{n-1}^2+x\,(1-x)\,\ell_{n-2}^2\right]^2} \tag{C.5}$$

and keeping only even powers of ℓ_{n-2} (and representing symmetrization over indices $(\mu_n \tau_n \mu_{n-1} \tau_{n-1})$ by $\otimes$),

$$= \int \mathrm{d}^4 \ell_{n-1} \, \mathrm{d}x$$
$$\frac{\otimes \left(\ell_{n-1}^{\nu_n} \ell_{n-1}^{\tau_n} \ell_{n-1}^{\nu_{n-1}} \ell_{n-1}^{\tau_{n-1}} \right) + x^2 \ell_{n-2}^{\nu_n} \ell_{n-2}^{\tau_n} \ell_{n-1}^{\nu_{n-1}} \ell_{n-1}^{\tau_{n-1}} + x \left(1-x\right) \ell_{n-1}^{\nu_n} \ell_{n-2}^{\tau_n} \ell_{n-1}^{\nu_{n-1}} \ell_{n-2}^{\tau_{n-1}}}{\left[\ell_{n-1}^2 + x \left(1-x\right) \ell_{n-2}^2 \right]^2}$$
$$+ \frac{x \left(1-x\right) \ell_{n-2}^{\nu_n} \ell_{n-1}^{\tau_n} \ell_{n-2}^{\nu_{n-1}} \ell_{n-1}^{\tau_{n-1}} + \left(1-x\right)^2 \ell_{n-1}^{\nu_n} \ell_{n-1}^{\tau_n} \ell_{n-2}^{\nu_{n-1}} \ell_{n-2}^{\tau_{n-1}} + x^2 \left(1-x\right)^2 \ell_{n-2}^{\nu_n} \ell_{n-2}^{\tau_n} \ell_{n-2}^{\nu_{n-1}} \ell_{n-2}^{\tau_{n-1}}}{\left[\ell_{n-1}^2 + x \left(1-x\right) \ell_{n-2}^2 \right]^2} \tag{C.6}$$

Symmetrizing the ℓ_{n-1} factors as before yields

$$= \int \mathrm{d}^4 \ell_{n-1} \, \mathrm{d}x$$
$$\frac{\frac{24 \ell_{n-1}^4}{d(d+2)} \otimes \left(g^{\nu_n \tau_n} g^{\nu_{n-1} \tau_{n-1}} \right) + x^2 \frac{\ell_{n-1}^2}{d} g^{\mu_{n-1} \nu_{n-1}} \ell_{n-2}^{\nu_n} \ell_{n-2}^{\tau_n} + x \left(1-x\right) \frac{\ell_{n-1}^2}{d} g^{\nu_n \nu_{n-1}} \ell_{n-2}^{\tau_n} \ell_{n-2}^{\tau_{n-1}}}{\left[\ell_{n-1}^2 + x \left(1-x\right) \ell_{n-2}^2 \right]^2}$$
$$+ \frac{x \left(1-x\right) \frac{\ell_{n-1}^2}{d} g^{\tau_n \nu_{n-1}} \ell_{n-2}^{\nu_n} \ell_{n-2}^{\tau_{n-1}} + \left(1-x\right)^2 \frac{\ell_{n-1}^2}{d} g^{\nu_n \tau_n} \ell_{n-2}^{\nu_{n-1}} \ell_{n-2}^{\tau_{n-1}} + x^2 \left(1-x\right)^2 \ell_{n-2}^{\nu_n} \ell_{n-2}^{\tau_n} \ell_{n-2}^{\nu_{n-1}} \ell_{n-2}^{\tau_{n-1}}}{\left[\ell_{n-1}^2 + x \left(1-x\right) \ell_{n-2}^2 \right]^2} . \tag{C.7}$$

Splitting the numerator up, we can once again apply the known integration expression with $\Delta^2 = -x\left(1-x\right)\ell_{n-2}^2$,

$$\int \frac{\mathrm{d}^d \ell}{\left(2\pi\right)^d} \frac{\ell^{2\beta}}{\left(\ell^2 - \Delta^2\right)^\alpha} = i \left(-1\right)^{\alpha+\beta} \frac{\Gamma\left(\beta + \frac{d}{2}\right) \Gamma\left(\alpha - \beta - \frac{d}{2}\right)}{\left(4\pi\right)^{\frac{d}{2}} \Gamma\left(\alpha\right) \Gamma\left(\frac{d}{2}\right)} \Delta^{2\left(\frac{d}{2} - \alpha + \beta\right)}, \tag{C.8}$$

yielding

$$\begin{aligned}
= \frac{i}{\left(4\pi\right)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \int_0^1 \mathrm{d}x \, & \frac{24}{d\left(d+2\right)} \otimes \left(g^{\nu_n \tau_n} g^{\nu_{n-1} \tau_{n-1}} \right) \Gamma\left(2 + \frac{d}{2}\right) \Gamma\left(-\frac{d}{2}\right) \Delta^d \\
& + \frac{x^2}{d} g^{\mu_{n-1} \nu_{n-1}} \ell_{n-2}^{\nu_n} \ell_{n-2}^{\tau_n} \Gamma\left(1 + \frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right) \Delta^{d-2} \\
& + \frac{x\left(1-x\right)}{d} g^{\nu_n \nu_{n-1}} \ell_{n-2}^{\tau_n} \ell_{n-2}^{\tau_{n-1}} \Gamma\left(1 + \frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right) \Delta^{d-2} \\
& + \frac{x\left(1-x\right)}{d} g^{\tau_n \nu_{n-1}} \ell_{n-2}^{\nu_n} \ell_{n-2}^{\tau_{n-1}} \Gamma\left(1 + \frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right) \Delta^{d-2}
\end{aligned}$$

$$+\frac{(1-x)^2}{d} g^{\nu_n\tau_n} \ell_{n-2}^{\nu_{n-1}} \ell_{n-2}^{\tau_{n-1}} \Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\Delta^{d-2}$$
$$+x^2(1-x)^2 \ell_{n-2}^{\nu_n}\ell_{n-2}^{\tau_n}\ell_{n-2}^{\nu_{n-1}}\ell_{n-2}^{\tau_{n-1}}\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\Delta^{d-4} \tag{C.9}$$

and substituting in Δ, we see that each term has ℓ^d dependence (upon symmetrization).

$$= \frac{i}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^1 \mathrm{d}x \, \frac{24}{d(d+2)} \otimes \left(g^{\nu_n\tau_n} g^{\nu_{n-1}\tau_{n-1}}\right)\Gamma\left(2+\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)(x(1-x))^{\frac{d}{2}}\ell_{n-2}^d$$
$$+\frac{1}{d} g^{\mu_{n-1}\nu_{n-1}} \ell_{n-2}^{\nu_n}\ell_{n-2}^{\tau_n}\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)x^{\frac{d}{2}+1}(1-x)^{\frac{d}{2}-1}\ell_{n-2}^{d-2}$$
$$+\frac{1}{d} g^{\nu_n\nu_{n-1}} \ell_{n-2}^{\tau_n}\ell_{n-2}^{\tau_{n-1}}\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)(x(1-x))^{\frac{d}{2}}\ell_{n-2}^{d-2}$$
$$+\frac{1}{d} g^{\tau_n\nu_{n-1}} \Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)(x(1-x))^{\frac{d}{2}}\ell_{n-2}^{\nu_n}\ell_{n-2}^{\tau_{n-1}}\ell_{n-2}^{d-2}$$
$$+\frac{1}{d} g^{\nu_n\tau_n} \Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)x^{\frac{d}{2}-1}(1-x)^{\frac{d}{2}+1}\ell_{n-2}^{\nu_{n-1}}\ell_{n-2}^{\tau_{n-1}}\ell_{n-2}^{d-2}$$
$$+\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)(x(1-x))^{\frac{d}{2}}\ell_{n-2}^{\nu_n}\ell_{n-2}^{\tau_n}\ell_{n-2}^{\nu_{n-1}}\ell_{n-2}^{\tau_{n-1}}\ell_{n-2}^{d-4} \tag{C.10}$$

where we exploit $\Gamma(1+x) = x\Gamma(x)$ to factor and cancel the gamma functions such that

$$= \frac{i\Gamma\left(-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 \mathrm{d}x \, \frac{24\left(1+\frac{d}{2}\right)\frac{d}{2}}{d(d+2)} \otimes \left(g^{\nu_n\tau_n} g^{\nu_{n-1}\tau_{n-1}}\right)(x(1-x))^{\frac{d}{2}}\ell_{n-2}^d$$
$$+\frac{\left(\frac{d}{2}\right)\left(-\frac{d}{2}\right)}{d} g^{\mu_{n-1}\nu_{n-1}} \ell_{n-2}^{\nu_n}\ell_{n-2}^{\tau_n}x^{\frac{d}{2}+1}(1-x)^{\frac{d}{2}-1}\ell_{n-2}^{d-2}$$
$$+\frac{\frac{d}{2}\left(-\frac{d}{2}\right)}{d} g^{\nu_n\nu_{n-1}}(x(1-x))^{\frac{d}{2}}\ell_{n-2}^{\tau_n}\ell_{n-2}^{\tau_{n-1}}\ell_{n-2}^{d-2}$$
$$+\frac{\frac{d}{2}\left(-\frac{d}{2}\right)}{d} g^{\tau_n\nu_{n-1}}(x(1-x))^{\frac{d}{2}}\ell_{n-2}^{\nu_n}\ell_{n-2}^{\tau_{n-1}}\ell_{n-2}^{d-2}$$
$$+\frac{\frac{d}{2}\left(-\frac{d}{2}\right)}{d} g^{\nu_n\tau_n}x^{\frac{d}{2}-1}(1-x)^{\frac{d}{2}+1}\ell_{n-2}^{\nu_{n-1}}\ell_{n-2}^{\tau_{n-1}}\ell_{n-2}^{d-2}$$
$$+\left(1-\frac{d}{2}\right)\left(-\frac{d}{2}\right)(x(1-x))^{\frac{d}{2}}\ell_{n-2}^{\nu_n}\ell_{n-2}^{\tau_n}\ell_{n-2}^{\nu_{n-1}}\ell_{n-2}^{\tau_{n-1}}\ell_{n-2}^{d-4} \tag{C.11}$$

Evaluating the Feynman integrals,

$$= \frac{i\Gamma\left(-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma(d+2)} \frac{24\left(1+\frac{d}{2}\right)\frac{d}{2}}{d(d+2)} \otimes \left(g^{\nu_n\tau_n} g^{\nu_{n-1}\tau_{n-1}}\right)\Gamma\left(\frac{d}{2}+1\right)^2\ell_{n-2}^d$$

$$
\begin{aligned}
&+ \frac{\left(\frac{d}{2}\right)\left(-\frac{d}{2}\right)}{d} g^{\mu_{n-1}\nu_{n-1}} \ell_{n-2}^{\nu_n} \ell_{n-2}^{\tau_n} \Gamma\left(\frac{d}{2}+2\right) \Gamma\left(\frac{d}{2}\right) \ell_{n-2}^{d-2} \\
&+ \frac{\frac{d}{2}\left(-\frac{d}{2}\right)}{d} g^{\nu_n \nu_{n-1}} \Gamma\left(\frac{d}{2}+1\right)^2 \ell_{n-2}^{\tau_n} \ell_{n-2}^{\tau_{n-1}} \ell_{n-2}^{d-2} \\
&+ \frac{\frac{d}{2}\left(-\frac{d}{2}\right)}{d} g^{\tau_n \nu_{n-1}} \Gamma\left(\frac{d}{2}+1\right)^2 \ell_{n-2}^{\nu_n} \ell_{n-2}^{\tau_{n-1}} \ell_{n-2}^{d-2} \\
&+ \frac{\frac{d}{2}\left(-\frac{d}{2}\right)}{d} g^{\nu_n \tau_n} \Gamma\left(\frac{d}{2}+2\right) \Gamma\left(\frac{d}{2}\right) \ell_{n-2}^{\nu_{n-1}} \ell_{n-2}^{\tau_{n-1}} \ell_{n-2}^{d-2} \\
&+ \left(1-\frac{d}{2}\right)\left(-\frac{d}{2}\right) \Gamma\left(\frac{d}{2}+1\right)^2 \ell_{n-2}^{\nu_n} \ell_{n-2}^{\tau_n} \ell_{n-2}^{\nu_{n-1}} \ell_{n-2}^{\tau_{n-1}} \ell_{n-2}^{d-4}.
\end{aligned}
\tag{C.12}
$$

D

Propagator Derivations

For QED, the momentum space photon propagator is given by

$$D^{\nu\rho} = \frac{-i}{k^2}\left(g^{\mu\nu} + \left(1 - \xi\frac{k^\mu k^\nu}{k^2}\right)\right).$$

This implies that it is the inverse of the quadratic term in the QED Lagrangian,

$$\mathcal{L} = A^\mu \underbrace{\left(-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right)}_{\text{quadratic term}} A^\nu. \tag{D.1}$$

Namely, we have

$$\left(-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right) D^{\nu\rho} = i\delta^\rho_\mu. \tag{D.2}$$

Proof.

$$\left(-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k_\mu k_\nu\right)\frac{-i}{k^2}\left(g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

$$= -\frac{i}{k^2}\left(-k^2\delta^\rho_\mu - (1-\xi)\,k^2 g_{\mu\nu}\frac{k^\nu k^\rho}{k^2} + \left(1-\frac{1}{\xi}\right)k_\mu k_\nu g^{\nu\rho} + \left(1-\frac{1}{\xi}\right)(1-\xi)\,\frac{k_\mu k_\nu k^\nu k^\rho}{k^2}\right)$$
$$= -\frac{i}{k^2}\left(-k^2\delta^\rho_\mu - (1-\xi)\,k_\mu k^\rho + \left(1-\frac{1}{\xi}\right)k_\mu k^\rho + \left(1-\frac{1}{\xi}\right)(1-\xi)\,k_\mu k^\rho\right)$$
$$= -\frac{i}{k^2}\left(-k^2\delta^\rho_\mu + \xi k_\mu k^\rho - \frac{1}{\xi}k_\mu k^\rho + \left(1-\xi-\frac{1}{\xi}+1\right)k_\mu k^\rho\right)$$
$$= -\frac{i}{k^2}\left(-k^2\delta^\rho_\mu + 2k_\mu k^\rho\right)$$
$$= -\frac{i}{k^2}\left(-k^2\delta^\rho_\mu\right)$$
$$= i\delta^\rho_\mu,$$

as desired. □

For ModMax, as the term quadratic in the quantum fields is entirely analogous up to a $\cosh\gamma$ factor and a background dependent term that can be absorbed through gauge choice

$$\mathcal{L}_{\text{ModMax}} = \frac{\cosh\gamma}{2}a_\nu\left(-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}-\tanh\gamma S_C\right)k^\mu k^\nu\right)a_\mu \text{ non-quadratic terms,} \tag{D.3}$$

the propagator is identical, with the addition of division by the $\cosh\gamma$ factor with

$$D^{\nu\rho} = \frac{1}{\cosh\gamma}\frac{-i}{k^2}\left(g^{\mu\nu} + \left(1-\left(\xi+\frac{1}{S_C\tanh\gamma}\right)\frac{k^\mu k^\nu}{k^2}\right)\right), \tag{D.4}$$

where throughout my thesis I have chosen ξ such that the propagator simplifies to

$$D^{\nu\rho} = \frac{1}{\cosh\gamma}\frac{-ig^{\mu\nu}}{k^2}. \tag{D.5}$$

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